Integration of Logarithmic Functions

Math 140: Calculus with Analytic Geometry

1 Introduction

We now study integrals that produce logarithmic functions. The key observation is

$$\frac{d}{dx}\ln|f(x)| = \frac{f'(x)}{f(x)}.$$

Thus, whenever an integrand has the form $\frac{f'(x)}{f(x)}$, the antiderivative is a logarithm.

This is another application of u-substitution.

2 The Logarithmic Rule

If f is differentiable and $f(x) \neq 0$, then

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C.$$

Example

Evaluate

$$\int \frac{2x}{x^2+1}\,dx.$$

Let $u = x^2 + 1$, so $du = 2x\,dx$. Then

$$\int \frac{2x}{x^2+1}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln(x^2+1) + C.$$

3 Definite Integrals

For definite integrals, we change the bounds.

Example

Evaluate

$$\int_0^2 \frac{2x}{x^2+1}\,dx.$$

Let $u = x^2 + 1$, so $du = 2x\,dx$. Then

$$x = 0 \Rightarrow u = 1, \qquad x = 2 \Rightarrow u = 5.$$

Thus,

$$\int_0^2 \frac{2x}{x^2+1}\,dx = \int_1^5 \frac{1}{u}\,du = \ln|u|\Big|_1^5 = \ln 5.$$

Example

Evaluate

$$\int_1^3 \frac{3}{3x+1}\,dx.$$

Let $u = 3x + 1$, so $du = 3\,dx$. Then

$$x = 1 \Rightarrow u = 4, \qquad x = 3 \Rightarrow u = 10.$$

Thus,

$$\int_1^3 \frac{3}{3x+1}\,dx = \int_4^{10} \frac{1}{u}\,du = \ln|u|\Big|_4^{10} = \ln 10 - \ln 4.$$

Example

Evaluate

$$\int_e^{e^2} \frac{1}{x\ln x}\,dx.$$

Let $u = \ln x$, so $du = \frac{1}{x}\,dx$. Then

$$x = e \Rightarrow u = 1, \qquad x = e^2 \Rightarrow u = 2.$$

Thus,

$$\int_e^{e^2} \frac{1}{x\ln x}\,dx = \int_1^2 \frac{1}{u}\,du = \ln|u|\Big|_1^2 = \ln 2.$$

4 Algebraic Preparation

In many integrals, the logarithmic structure is not immediately visible. The goal is to rewrite the integrand so that it matches the form $\frac{f'(x)}{f(x)}$.

4.1 Long Division

If the degree of the numerator is greater than or equal to the degree of the denominator, we first perform polynomial division. This separates the integrand into a polynomial term and a simpler rational function.

Example

Evaluate

$$\int_0^2 \frac{x^2}{x+1}\,dx.$$

Divide:

$$\frac{x^2}{x+1} = x - 1 + \frac{1}{x+1}.$$

Thus,

$$\int_0^2 \frac{x^2}{x+1}\,dx = \int_0^2 (x-1)\,dx + \int_0^2 \frac{1}{x+1}\,dx.$$

$$\int_0^2 (x-1)\,dx = \left(\frac{x^2}{2} - x\right)\Big|_0^2 = 0,$$

$$\int_0^2 \frac{1}{x+1}\,dx = \ln|x+1|\Big|_0^2 = \ln 3.$$

Therefore,

$$\int_0^2 \frac{x^2}{x+1}\,dx = \ln 3.$$

4.2 Partial Fractions (Linear Factors)

If the denominator factors into a product of distinct linear terms, we write

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}.$$

Example

Evaluate

$$\int_2^3 \frac{1}{x^2-1}\,dx.$$

Factor:

$$x^2 - 1 = (x-1)(x+1).$$

Write

$$\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}.$$

Multiply:

$$1 = A(x+1) + B(x-1).$$

$$x = 1 \Rightarrow A = \frac{1}{2}, \qquad x = -1 \Rightarrow B = -\frac{1}{2}.$$

Thus,

$$\frac{1}{x^2 - 1} = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right).$$

$$\int_2^3 \frac{1}{x^2-1}\,dx = \frac{1}{2}\int_2^3 \left(\frac{1}{x-1} - \frac{1}{x+1}\right)dx$$

$$= \frac{1}{2}\left(\ln|x-1| - \ln|x+1|\right)\Big|_2^3.$$

5 Trigonometric Examples

Example

Evaluate

$$\int_0^{\pi/4} \tan x\,dx.$$

Write $\tan x = \frac{\sin x}{\cos x}$ and let $u = \cos x$, $du = -\sin x\,dx$.

$$x = 0 \Rightarrow u = 1, \qquad x = \frac{\pi}{4} \Rightarrow u = \frac{\sqrt{2}}{2}.$$

$$\int_0^{\pi/4} \tan x\,dx = -\int_1^{\sqrt{2}/2} \frac{1}{u}\,du = -\ln|u|\Big|_1^{\sqrt{2}/2} = \ln(\sqrt{2}).$$

Example

Evaluate

$$\int_0^{\pi/3} \sec x\,dx.$$

Rewrite and let $u = \sec x + \tan x$.

$$x = 0 \Rightarrow u = 1, \qquad x = \frac{\pi}{3} \Rightarrow u = 2 + \sqrt{3}.$$

$$\int_0^{\pi/3} \sec x\,dx = \int_1^{2+\sqrt{3}} \frac{1}{u}\,du = \ln|u|\Big|_1^{2+\sqrt{3}} = \ln(2+\sqrt{3}).$$

6 Summary

Logarithmic integrals arise when the integrand has the form

$$\frac{f'(x)}{f(x)}.$$

We use substitution, algebra, and rewriting to reveal this structure.

For definite integrals, we always change the bounds and evaluate using

$$F(x)\Big|_a^b.$$