# Inverse Trigonometric Integrals

Math 140: Calculus with Analytic Geometry

## 1 Introduction

We now study integrals whose antiderivatives involve inverse trigonometric functions. The most important basic forms are

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C, \qquad \int \frac{1}{1+x^2}\,dx = \arctan(x) + C,$$

$$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \operatorname{arcsec}(|x|) + C.$$

Most integrals are not initially in these forms. Instead, we use algebra and substitution to rewrite the integrand so that one of these patterns appears.

## 2 Rewriting the Integrand

### Example

Evaluate

$$\int \frac{x+2}{\sqrt{4-x^2}}\,dx.$$

Split the integral:

$$= \int \frac{x}{\sqrt{4-x^2}}\,dx + \int \frac{2}{\sqrt{4-x^2}}\,dx.$$

For the first integral, let $u = 4 - x^2$, so $du = -2x\,dx$:

$$= -\sqrt{4-x^2}.$$

For the second, factor:

$$\int \frac{2}{\sqrt{4-x^2}}\,dx = \int \frac{1}{\sqrt{1-(x/2)^2}}\,dx.$$

Let $u = \frac{x}{2}$, so $dx = 2\,du$:

$$= 2\arcsin(u) = 2\arcsin\left(\frac{x}{2}\right).$$

Thus,

$$= -\sqrt{4-x^2} + 2\arcsin\left(\frac{x}{2}\right) + C.$$

**Example**

Evaluate

$$\int_0^1 \frac{x+2}{\sqrt{4-x^2}}\,dx.$$

$$= \left(-\sqrt{4-x^2} + 2\arcsin\left(\frac{x}{2}\right)\right)\Big|_0^1.$$

# 3 Completing the Square

**Example**

Evaluate

$$\int \frac{1}{x^2-4x+7}\,dx.$$

Complete the square:

$$x^2-4x+7=(x-2)^2+3.$$

Let $u=\frac{x-2}{\sqrt{3}}$, so $dx=\sqrt{3}\,du$:

$$= \int \frac{1}{1+u^2}\,du = \arctan(u)+C.$$

$$= \arctan\left(\frac{x-2}{\sqrt{3}}\right)+C.$$

**Example**

Evaluate

$$\int_0^2 \frac{1}{x^2-4x+7}\,dx.$$

$$= \arctan\left(\frac{x-2}{\sqrt{3}}\right)\Big|_0^2.$$

# 4 Square Root Expressions

**Example**

Evaluate

$$\int \frac{1}{\sqrt{3x-x^2}}\,dx.$$

Rewrite:

$$3x-x^2=\frac{9}{4}-\left(x-\frac{3}{2}\right)^2.$$

$$= \int \frac{1}{\sqrt{1 - \left(\frac{x-\frac{3}{2}}{\frac{3}{2}}\right)^2}}\, dx.$$

Let

$$u = \frac{x - \frac{3}{2}}{\frac{3}{2}}, \qquad dx = \frac{3}{2}\, du.$$

$$= \frac{3}{2} \int \frac{1}{\sqrt{1-u^2}}\, du = \frac{3}{2} \arcsin(u) + C.$$

$$= \frac{3}{2} \arcsin\left(\frac{2x-3}{3}\right) + C.$$

**Example**

Evaluate

$$\int_0^3 \frac{1}{\sqrt{3x - x^2}}\, dx.$$

$$= \frac{3}{2} \arcsin\left(\frac{2x-3}{3}\right)\Big|_0^3.$$

# 5 Exponential Substitution and Arcsec

**Example**

Evaluate

$$\int \frac{1}{\sqrt{e^{2x} - 1}}\, dx.$$

Let $u = e^x$, so $dx = \frac{du}{u}$:

$$= \int \frac{1}{u\sqrt{u^2 - 1}}\, du = \operatorname{arcsec}(u) + C.$$

$$= \operatorname{arcsec}(e^x) + C.$$

**Example**

Evaluate

$$\int_0^{\ln 2} \frac{1}{\sqrt{e^{2x} - 1}}\, dx.$$

Let $u = e^x$:

$$x = 0 \Rightarrow u = 1, \qquad x = \ln 2 \Rightarrow u = 2.$$

$$= \int_1^2 \frac{1}{u\sqrt{u^2 - 1}}\, du = \operatorname{arcsec}(u)\Big|_1^2.$$

## 6 Summary

Inverse trigonometric integrals arise after rewriting an integrand into one of the forms

$$\frac{1}{\sqrt{1-x^2}}, \quad \frac{1}{1+x^2}, \quad \frac{1}{x\sqrt{x^2-1}}.$$

The key idea is not memorizing formulas, but rewriting the integrand so that these patterns appear.