U-Substitution

Math 140: Calculus with Analytic Geometry

1 Introduction

Now that we have the Fundamental Theorem of Calculus, the problem of finding area is reduced to the problem of finding antiderivatives. Thus, our next goal is to develop methods for evaluating more complicated indefinite and definite integrals.

So far, we have discussed the basic properties of indefinite integrals and the power rule. In this lecture, we introduce *u-substitution*. This method is motivated by the chain rule. The main idea is that some integrals are difficult in the variable x, but become much easier after replacing a complicated expression by a new variable u.

2 Motivation from the Chain Rule

Recall the chain rule. If $F'(x) = f(x)$, then

$$\frac{d}{dx}F(g(x)) = f(g(x))g'(x).$$

This means that if an integrand has the form

$$f(g(x))g'(x),$$

then its antiderivative is

$$F(g(x)).$$

Thus, integration can sometimes be viewed as reversing the chain rule.

Example

Evaluate

$$\int 2x(x^2+1)^3\,dx.$$

The quantity x^2+1 appears inside the power, and its derivative is $2x$. This suggests the substitution

$$u = x^2 + 1.$$

Then

$$du = 2x\,dx.$$

Therefore,

$$\int 2x(x^2+1)^3\,dx = \int u^3\,du = \frac{u^4}{4} + C.$$

Substituting back gives

$$\int 2x(x^2+1)^3\,dx = \frac{(x^2+1)^4}{4} + C.$$

This example illustrates the basic substitution rule.

Substitution Rule

If

$$u = g(x) \qquad \text{and} \qquad du = g'(x)\,dx,$$

then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

After evaluating the integral in terms of u, we substitute back to write the final answer in terms of x.

3 Examples of U-Substitution

In each of the following examples, we look for an inside function whose derivative also appears, perhaps up to a constant factor, elsewhere in the integrand.

Example

Evaluate

$$\int (3x-1)^5\,dx.$$

Let

$$u = 3x - 1.$$

Then

$$du = 3\,dx, \qquad dx = \frac{1}{3}\,du.$$

Thus,

$$\int (3x-1)^5\,dx = \int u^5 \cdot \frac{1}{3}\,du = \frac{1}{3}\int u^5\,du = \frac{1}{3}\cdot\frac{u^6}{6} + C = \frac{u^6}{18} + C.$$

Substituting back, we obtain

$$\int (3x-1)^5\,dx = \frac{(3x-1)^6}{18} + C.$$

Example

Evaluate

$$\int x\sqrt{x^2+4}\,dx.$$

Let

$$u = x^2 + 4.$$

Then

$$du = 2x\,dx, \qquad x\,dx = \frac{1}{2}\,du.$$

Therefore,

$$\int x\sqrt{x^2+4}\,dx = \frac{1}{2}\int u^{1/2}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C = \frac{1}{3}u^{3/2} + C.$$

Hence,

$$\int x\sqrt{x^2+4}\,dx = \frac{1}{3}(x^2+4)^{3/2} + C.$$

Example

Evaluate

$$\int \frac{x}{x^2+1}\,dx.$$

This example is slightly less obvious because the antiderivative is not obtained by the power rule. However, the denominator contains x^2+1, whose derivative is $2x$. This suggests

$$u = x^2 + 1, \qquad du = 2x\,dx, \qquad x\,dx = \frac{1}{2}\,du.$$

Thus,

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C.$$

Substituting back gives

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1) + C.$$

4 Definite Integrals and Change of Bounds

When we apply substitution to a definite integral, we will always change the bounds of integration. This keeps the work consistent and makes it clear that the variable has really changed.

If

$$u = g(x),$$

then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Example

Evaluate

$$\int_0^2 x(x^2+1)^3\,dx.$$

Let

$$u = x^2 + 1.$$

Then

$$du = 2x\,dx, \qquad x\,dx = \frac{1}{2}\,du.$$

Now change the bounds:

$$x = 0 \Rightarrow u = 1, \qquad x = 2 \Rightarrow u = 5.$$

Therefore,

$$\int_0^2 x(x^2+1)^3\,dx = \frac{1}{2}\int_1^5 u^3\,du = \frac{1}{2}\left(\frac{u^4}{4}\right)\Big|_1^5 = \frac{1}{8}(5^4 - 1^4).$$

Example

Evaluate

$$\int_0^2 x(x^2+1)\,dx.$$

Let

$$u = x^2 + 1, \qquad du = 2x\,dx, \qquad x\,dx = \frac{1}{2}\,du.$$

When $x = 0$, we have $u = 1$, and when $x = 2$, we have $u = 5$. Thus,

$$\int_0^2 x(x^2+1)\,dx = \frac{1}{2}\int_1^5 u\,du = \frac{1}{2}\left(\frac{u^2}{2}\right)\Big|_1^5 = \frac{1}{4}(25 - 1) = 6.$$

This substitution turns the original integral into a much simpler one. The geometry of the region also becomes easier to understand.

A geometric picture

The integral

$$\int_0^2 x(x^2+1)\,dx$$

represents the area under the curve $y = x(x^2+1)$ from $x = 0$ to $x = 2$. After the substitution $u = x^2 + 1$, the same numerical value is written as

$$\frac{1}{2}\int_1^5 u\,du.$$

Thus, the original integral is transformed into one over a simpler interval with a simpler integrand. In the u-variable, we are finding the area under the line $y = \frac{1}{2}u$ on the interval $[1, 5]$.

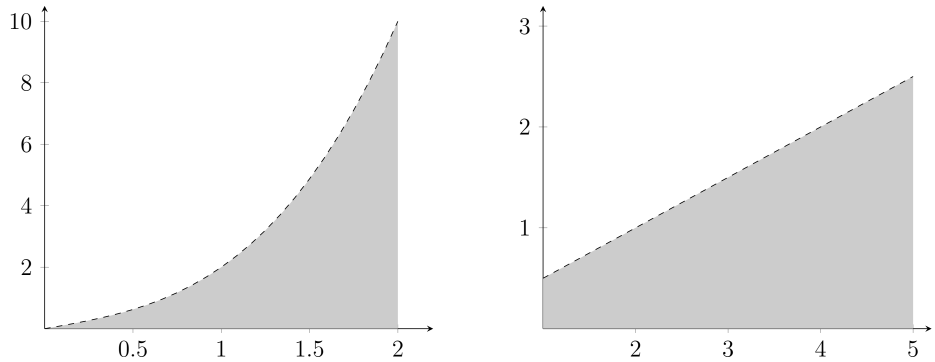

Figure 1: On the left is the region for $\int_0^2 x(x^2+1)\,dx$. On the right is the simpler region for $\frac{1}{2}\int_1^5 u\,du$ obtained after the substitution $u = x^2 + 1$.

5 Summary

U-substitution is motivated by the chain rule. It is a method for rewriting an integral in a new variable so that the antiderivative becomes easier to find.

In the examples from this lecture, substitution turns a more complicated integral into one of the basic forms we already know how to integrate. For definite integrals, we change the bounds when we change variables. This keeps the computation in the new variable from beginning to end.

In later lectures, we will apply this same idea to more complicated functions involving logarithms, trigonometric functions, and inverse trigonometric functions.