Trigonometric Functions

Math 140: Calculus with Analytic Geometry

Key Topics

- Trigonometric functions from right triangles
- The Pythagorean identity
- Extension to the unit circle
- Quadrants, symmetry, and periodicity
- Graphs of sine, cosine, and tangent

1 Right Triangle Definitions

We begin by defining trigonometric functions using a right triangle. Let θ be an acute angle in a right triangle.

Definition. *The trigonometric functions* $\sin(\theta)$, $\cos(\theta)$, *and* $\tan(\theta)$ *are defined by*

$$\sin(\theta) = \frac{O}{H}, \qquad \cos(\theta) = \frac{A}{H}, \qquad \tan(\theta) = \frac{O}{A},$$

where H is the length of the hypotenuse, O is the length of the side opposite θ, and A is the length of the side adjacent to θ. Note that $\csc(\theta)$, $\sec(\theta)$, *and* $\cot(\theta)$ *are the reciprocals of* $\sin(\theta)$, $\cos(\theta)$, *and* $\tan(\theta)$, *respectively.*

Figure 1 shows a right triangle illustrating these definitions.

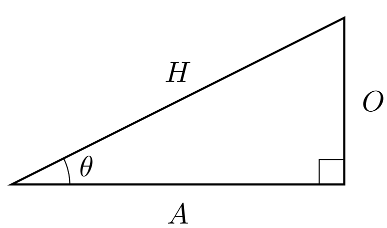

Figure 1: A right triangle defining sine, cosine, and tangent as ratios of side lengths.

2 The Pythagorean Identity

Using the right triangle in Figure 1, the Pythagorean theorem gives

$$A^2 + O^2 = H^2.$$

Dividing both sides by H^2 yields

$$\left(\frac{A}{H}\right)^2 + \left(\frac{O}{H}\right)^2 = 1.$$

By definition,

$$\cos^2(\theta) + \sin^2(\theta) = 1.$$

Remark. *This identity will be used frequently when simplifying expressions involving trigonometric functions in calculus. Note that there are other forms for the Pythagorean identity; for example,*

$$1 + \tan^2(\theta) = \sec^2(\theta)$$

and

$$\cot^2(\theta) + 1 = \csc^2(\theta).$$

3 Extending Sine and Cosine to the Unit Circle

The right-triangle definitions apply only to acute angles. To define sine and cosine for all real numbers, we use the unit circle.

Definition. *The* unit circle *is the circle of radius* 1 *centered at the origin. For an angle θ in standard position,* $\cos(\theta)$ *is the x-coordinate and* $\sin(\theta)$ *is the y-coordinate of the point where the terminal side of θ intersects the unit circle.*

Figure 2 shows how the right triangle from the first quadrant is embedded in the unit circle.

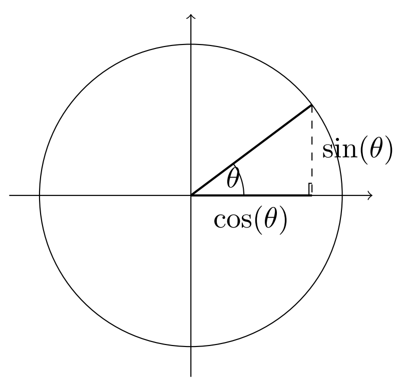

Figure 2: The right triangle associated with θ embedded in the unit circle.

4 Quadrants, Symmetry, and Periodicity

When θ lies in the second quadrant, the angle is no longer acute. Instead, we use the reference angle $\alpha = \pi - \theta$, which is acute. The vertical leg of the triangle remains the same length, while the horizontal leg changes sign.

This leads to the identities

$$\cos(\pi - \theta) = -\cos(\theta), \qquad \sin(\pi - \theta) = \sin(\theta).$$

Once sine and cosine are defined in the first and second quadrants, symmetry of the unit circle determines their values in the remaining quadrants. For example,

$$\cos(-\theta) = \cos(\theta), \qquad \sin(-\theta) = -\sin(\theta),$$

and

$$\cos(\theta + 2\pi) = \cos(\theta), \qquad \sin(\theta + 2\pi) = \sin(\theta).$$

5 Graphs of Sine, Cosine, and Tangent

The graphs of sine and cosine are obtained by plotting their unit-circle values as functions of θ.

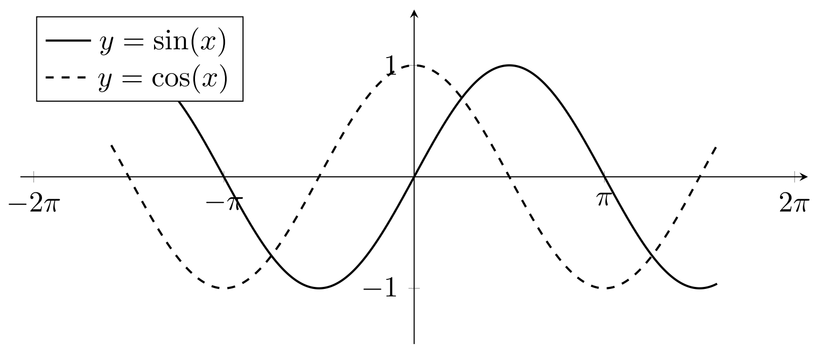

Figure 3: Graphs of the sine and cosine functions over two full periods.

Figure 3 reflects the periodic behavior implied by the unit-circle definition. The tangent function is defined by

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)},$$

whenever $\cos(\theta) \neq 0$.

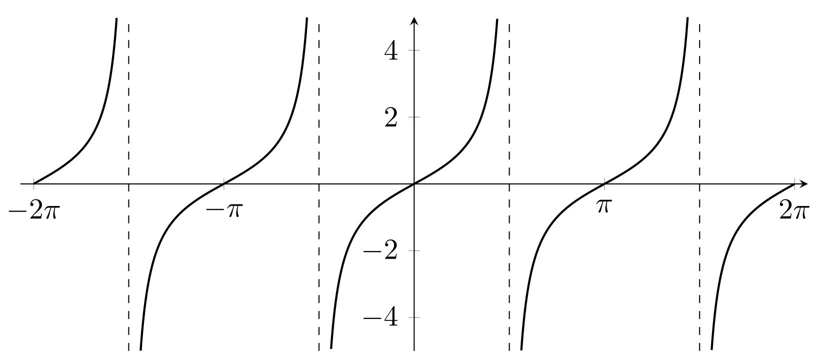

Figure 4: The graph of $y = \tan(x)$, showing vertical asymptotes where $\cos(x) = 0$.

6 Inverse Trig Functions

The functions of sin, cos, and tan map angles to ratios on the right triangle. Therefore, the inverse functions arcsin, arccos, and arctan map ratios on the right triangle to angles. For example, $\sin(\pi/6) = 1/2$; so, $\arcsin(1/2) = \pi/6$.

Note that the graphs of sin, cos, and tan do not pass the horizontal line test. Therefore, these functions do not have inverses defined on their entire domain. However, we can restrict their domains to allow for their inverses to be well-defined. In particular, we restrict the domain of sin to $[-\pi/2, \pi/2]$, the domain of cos to $[0, \pi]$, and the domain of tan to $(-\pi/2, \pi/2)$.

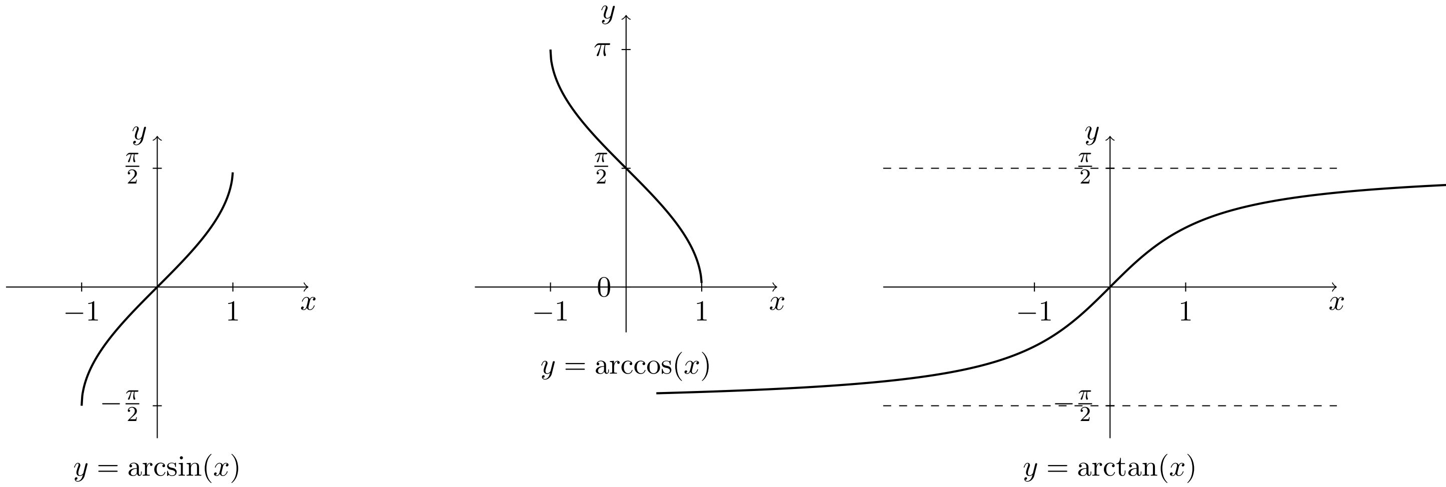

Figure 5: Graphs of $y = \arcsin(x)$, $y = \arccos(x)$, and $y = \arctan(x)$.

7 Why This Matters for Calculus

- Trigonometric functions appear frequently in limits, derivatives, and integrals.
- The unit-circle definition explains identities used in differentiation.
- Understanding periodicity and symmetry simplifies many calculus computations.