Derivatives of Trigonometric Functions

Math 140: Calculus with Analytic Geometry
Week 5

Key Topics

- Special trigonometric limits and their geometric meaning
- Derivatives of $\sin(x)$ and $\cos(x)$ using the limit definition
- Use of trigonometric identities in differentiation
- Derivatives of $\tan(x)$, $\cot(x)$, $\sec(x)$, and $\csc(x)$ using the quotient rule
- Tangent lines to trigonometric functions
- Geometric interpretation of trigonometric derivatives

1 Special Trigonometric Limits

$$\lim_{x\to 0}\frac{\sin(x)}{x}=1, \qquad \lim_{x\to 0}\frac{\cos(x)-1}{x}=0.$$

These limits are established geometrically using the unit circle and will be used to derive the derivatives of trigonometric functions from first principles.

2 Derivatives of $\sin(x)$ and $\cos(x)$sin(x) and cos(x)

Theorem 2.1.

$$\frac{d}{dx}\big(\sin(x)\big)=\cos(x).$$

Proof. Fix $a\in\mathbb{R}$. Using the limit definition of the derivative,

$$\frac{d}{dx}\big(\sin(x)\big)\Big|_{x=a}=\lim_{h\to 0}\frac{\sin(a+h)-\sin(a)}{h}.$$

Using the identity

$$\sin(a+h)=\sin(a)\cos(h)+\cos(a)\sin(h),$$

we obtain

$$\frac{\sin(a+h)-\sin(a)}{h}=\sin(a)\frac{\cos(h)-1}{h}+\cos(a)\frac{\sin(h)}{h}.$$

Taking limits and using the special trigonometric limits,

$$\frac{d}{dx}\big(\sin(x)\big)\Big|_{x=a}=\sin(a)\cdot 0+\cos(a)\cdot 1=\cos(a).$$

Since this holds for every real number a, it follows that

$$\frac{d}{dx}\big(\sin(x)\big)=\cos(x).$$

□

Theorem 2.2.

$$\frac{d}{dx}\big(\cos(x)\big) = -\sin(x).$$

Proof. Fix $a \in \mathbb{R}$. Using the limit definition,

$$\frac{d}{dx}\big(\cos(x)\big)\bigg|_{x=a} = \lim_{h\to 0} \frac{\cos(a+h) - \cos(a)}{h}.$$

Using the identity

$$\cos(a+h) = \cos(a)\cos(h) - \sin(a)\sin(h),$$

we obtain

$$\frac{\cos(a+h) - \cos(a)}{h} = \cos(a)\frac{\cos(h) - 1}{h} - \sin(a)\frac{\sin(h)}{h}.$$

Taking limits gives

$$\frac{d}{dx}\big(\cos(x)\big)\bigg|_{x=a} = \cos(a)\cdot 0 - \sin(a)\cdot 1 = -\sin(a).$$

Thus,

$$\frac{d}{dx}\big(\cos(x)\big) = -\sin(x).$$

□

3 Derivatives of the Remaining Trigonometric Functions

Theorem 3.1.

$$\frac{d}{dx}\big(\tan(x)\big) = \sec^2(x).$$

Proof. Since $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$, the quotient rule gives

$$\frac{d}{dx}\big(\tan(x)\big) = \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \sec^2(x).$$

□

Theorem 3.2.

$$\frac{d}{dx}\big(\cot(x)\big) = -\csc^2(x).$$

Proof. Since $\cot(x) = \dfrac{\cos(x)}{\sin(x)}$,

$$\frac{d}{dx}\big(\cot(x)\big) = \frac{-\sin(x)\sin(x) - \cos(x)\cos(x)}{\sin^2(x)} = -\csc^2(x).$$

□

Theorem 3.3.

$$\frac{d}{dx}\big(\sec(x)\big) = \sec(x)\tan(x), \qquad \frac{d}{dx}\big(\csc(x)\big) = -\csc(x)\cot(x).$$

Proof. Write $\sec(x) = 1/\cos(x)$ and $\csc(x) = 1/\sin(x)$ and apply the quotient rule. □

4 Examples

Example 4.1. *Differentiate* $f(x) = x^2 \sin(x)$.

$$f'(x) = 2x\sin(x) + x^2\cos(x).$$

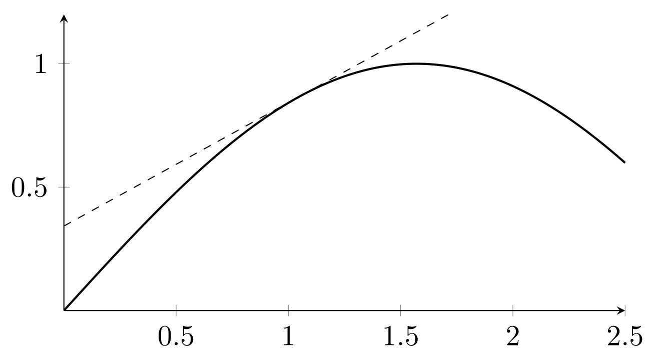
Example 4.2. *Differentiate* $g(x) = \dfrac{\cos(x)}{x}$.

$$g'(x) = \frac{-x\sin(x) - \cos(x)}{x^2}.$$

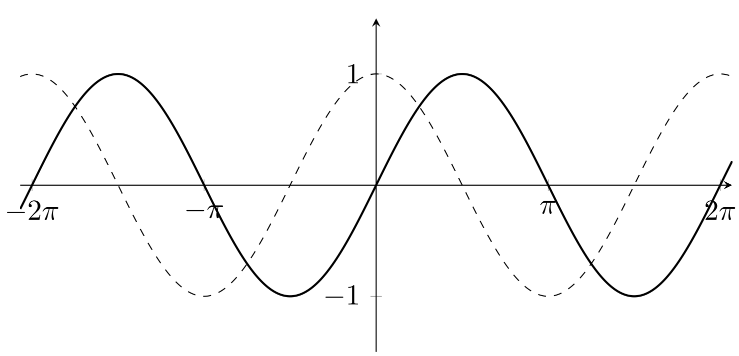
Example 4.3. *Find the equation of the tangent line to* $y = \sin(x)$ *at* $x = \dfrac{\pi}{3}$.

The slope is $\cos(\pi/3) = \dfrac{1}{2}$ and the point is $\left(\dfrac{\pi}{3}, \dfrac{\sqrt{3}}{2}\right)$. Thus,

$$y - \frac{\sqrt{3}}{2} = \frac{1}{2}\left(x - \frac{\pi}{3}\right).$$



5 Geometric Interpretation



Remark 5.1. *The graph of* $y = \cos(x)$ *represents the slope of* $y = \sin(x)$ *at each point. Where* $\cos(x) > 0$*, the graph of* $\sin(x)$ *is increasing; where* $\cos(x) < 0$*, the graph of* $\sin(x)$ *is decreasing. Likewise, the graph of* $y = -\sin(x)$ *gives the slope of* $y = \cos(x)$.

6 Why This Matters for Calculus

Trigonometric functions arise naturally when modeling periodic and oscillatory behavior, but their importance in calculus goes beyond applications.

- The derivatives of $\sin(x)$ and $\cos(x)$ are the first examples of functions whose derivatives do not simply reduce degree, as with polynomials. Instead, differentiation reveals a repeating structure: differentiating sine produces cosine, and differentiating cosine produces negative sine.
- The limit-based derivations of $\dfrac{d}{dx}\big(\sin(x)\big)$ and $\dfrac{d}{dx}\big(\cos(x)\big)$ show how geometric information from the unit circle is encoded algebraically through derivatives.
- Using the quotient rule to derive the remaining trigonometric derivatives reinforces how new rules in calculus are built from previously established ones, rather than introduced in isolation.
- Trigonometric derivatives combine naturally with the power rule, product rule, and quotient rule, allowing us to differentiate a wide class of functions encountered in later sections.
- These results prepare us for studying higher–order derivatives, the chain rule, and applications involving motion, waves, and oscillations in subsequent chapters.