

# Related Rates

Math 140: Calculus with Analytic Geometry

## Key Topics

- Rates of change with respect to time
- Implicit differentiation in applied settings
- Geometric relationships between changing quantities
- Volumes, angles, and multi-step related rates problems

## 1 Motivation and Strategy

In many applications, quantities change with respect to time.

For example:

- The height of a ladder decreases as it slides down a wall.
- The radius of a balloon increases as air is pumped into it.
- The length of a shadow changes as a person walks away from a light.

In each case, multiple quantities change simultaneously, and those changes are connected through a geometric relationship.

If a quantity  $x$  depends on time  $t$ , we denote its rate of change by

$$\frac{dx}{dt}.$$

Even when no explicit formula for  $x(t)$  is given, we treat each changing quantity as a differentiable function of time.

## General Strategy for Related Rates Problems

- Step 1.** Draw a diagram and assign variables.
  - Step 2.** Write an equation relating the variables.
  - Step 3.** Differentiate implicitly with respect to time  $t$ .
  - Step 4.** Substitute numerical values and solve for the desired rate.
- Remark 1.1. Always substitute numerical values *after* differentiating.

## 2 Basic Examples

### Example 2.1: Expanding Circle

The radius of a circle increases at 3 cm/s. How fast is the area increasing when  $r = 5$  cm?

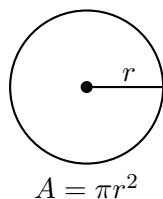
$$A = \pi r^2.$$

Differentiate:

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}.$$

Substitute:

$$\frac{dA}{dt} = 2\pi(5)(3) = 30\pi.$$



### Example 2.2: Sliding Ladder

A 10 ft ladder leans against a wall. The bottom slides away at 2 ft/s. How fast is the top sliding down when the bottom is 6 ft from the wall?

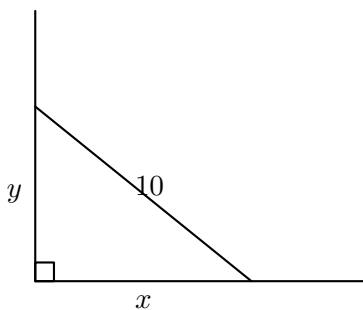
$$x^2 + y^2 = 100.$$

Differentiate:

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \quad \Rightarrow \quad \frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt}.$$

When  $x = 6$ , we find  $y = 8$ , so

$$\frac{dy}{dt} = -\frac{6}{8}(2) = -\frac{3}{2}.$$



### Example 2.3: Moving Shadow

A 6 ft tall person walks away from a 15 ft pole at 4 ft/s. How fast is the tip of the shadow moving when the person is 10 ft from the pole?

Using similar triangles:

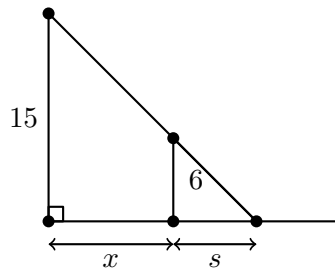
$$\frac{15}{x+s} = \frac{6}{s} \quad \Rightarrow \quad 15s = 6(x+s) \quad \Rightarrow \quad s = \frac{2}{3}x.$$

Differentiate:

$$\frac{ds}{dt} = \frac{2}{3} \frac{dx}{dt} = \frac{8}{3}.$$

The tip moves at

$$\frac{d}{dt}(x+s) = \frac{dx}{dt} + \frac{ds}{dt} = 4 + \frac{8}{3} = \frac{20}{3}.$$



### Example 2.4: Changing Rectangle

A rectangle has length  $L$  and width  $W$ . Suppose  $L$  increases at 3 cm/s while  $W$  decreases at 1 cm/s. How fast is the area changing when  $L = 10$  cm and  $W = 4$  cm?

$$A = LW.$$

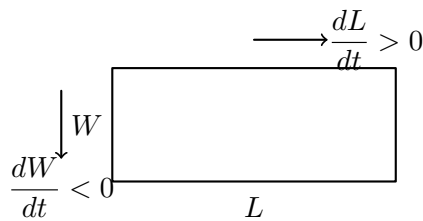
Differentiate:

$$\frac{dA}{dt} = L \frac{dW}{dt} + W \frac{dL}{dt}.$$

Substitute:

$$\frac{dA}{dt} = 10(-1) + 4(3) = -10 + 12 = 2.$$

Thus, the area is increasing at 2 cm<sup>2</sup>/s.



### 3 Advanced Examples

#### Example 3.1: Filling a Cone

Water flows into a cone at  $5 \text{ cm}^3/\text{s}$ . The cone has height 10 cm and radius 5 cm. How fast is the water level rising when  $h = 4 \text{ cm}$ ?

$$V = \frac{1}{3}\pi r^2 h.$$

From similar triangles:

$$\frac{r}{h} = \frac{5}{10} = \frac{1}{2} \Rightarrow r = \frac{h}{2}.$$

Substitute:

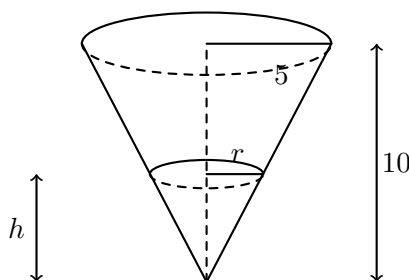
$$V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3.$$

Differentiate:

$$\frac{dV}{dt} = \frac{\pi}{4}h^2 \frac{dh}{dt}.$$

At  $h = 4$ :

$$5 = \frac{\pi}{4}(16) \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{5}{4\pi}.$$



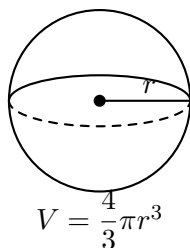
### Example 3.2: Expanding Sphere

Air is pumped into a spherical balloon at  $10 \text{ cm}^3/\text{s}$ . How fast is the radius increasing when  $r = 3 \text{ cm}$ ?

$$V = \frac{4}{3}\pi r^3 \quad \Rightarrow \quad \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

Substitute:

$$10 = 4\pi(9) \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{5}{18\pi}.$$



### Example 3.3: Angle of Elevation

A person stands 50 ft from a building. If the height  $y$  of a point on the building increases at  $6 \text{ ft/s}$ , how fast is the angle  $\theta$  increasing when  $\theta = \frac{\pi}{4}$ ?

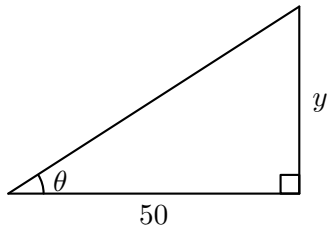
$$\tan(\theta) = \frac{y}{50}.$$

Differentiate:

$$\sec^2(\theta) \frac{d\theta}{dt} = \frac{1}{50} \frac{dy}{dt}.$$

At  $\theta = \frac{\pi}{4}$ ,  $\sec^2(\theta) = 2$ , so

$$2 \frac{d\theta}{dt} = \frac{6}{50} \Rightarrow \frac{d\theta}{dt} = \frac{3}{50}.$$



### Example 3.4: Two Cars

Car A travels east at 40 mph. Car B travels north at 30 mph. How fast is the distance between them increasing after 2 hours?

After 2 hours:

$$x = 80, \quad y = 60, \quad z^2 = x^2 + y^2.$$

Differentiate:

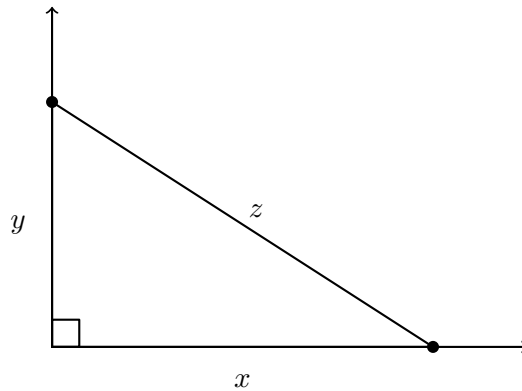
$$2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}.$$

Compute  $z$ :

$$z = \sqrt{80^2 + 60^2} = 100.$$

Substitute:

$$2(100) \frac{dz}{dt} = 2(80)(40) + 2(60)(30) \Rightarrow \frac{dz}{dt} = 50.$$



## 4 Why This Matters for Calculus

Related rates problems demonstrate how derivatives model real-time change.

- They combine geometry, algebra, and implicit differentiation.
- They reinforce the chain rule in applied settings.
- They prepare us for optimization and modeling.
- They connect abstract derivative rules to physical meaning.