Product and Quotient Rules

Math 140: Calculus with Analytic Geometry

Key Topics

- Derivative Operator d/dx.
- Product Rule
- Power Rule
- Quotient Rule
- Efficient differentiation of rational functions
- Common pitfalls

1 Motivation

Many functions encountered in applications are formed by multiplying or dividing simpler functions. The linearity rules from the previous lecture are not sufficient for these cases, so we introduce two new derivative rules. We also introduce the concept of the derivative operator, which symbolizes the action of taking a derivative. We denote this operator as follows

$$\frac{d}{dx}f(x) = f'(x).$$

2 Product Rule

Last lecture, we proved the product rule. In particular, if $f(x) = u(x)v(x)$, then

$$f'(x) = u'(x)v(x) + u(x)v'(x).$$

Using the derivative operator, we have

$$\frac{d}{dx}f(x) = v(x) \cdot \frac{d}{dx}u(x) + u(x) \cdot \frac{d}{dx}v(x).$$

3 Power Rule

Last lecture, we used the product rule to derive the power rule for positive integers; in particular, we showed that if $f(x) = x^n$, where n is a postive integer, then $f'(x) = nx^{n-1}$. Moreover, we eluded to the fact that power rule holds for all real numbers, which we will prove later in the course. In fact, we have seen that the power rule holds for $n = -1$. As a reminder, recall the following limit

$$\begin{aligned}
\lim_{h\to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h} &= \lim_{h\to 0} \frac{\frac{a-(a+h)}{a(a+h)}}{h} \\
&= \lim_{h\to 0} \frac{\frac{-h}{a(a+h)}}{h} \\
&= \lim_{h\to 0} \frac{-1}{a(a+h)} = -\frac{1}{a^2}.
\end{aligned}$$

Hence,

$$\frac{d}{dx}x^{-1} = -1x^{-2} = -\frac{1}{x^2}.$$

This derivative rule can also be viewed as an example of the quotient rule.

4 Quotient Rule

Theorem 4.1 (Quotient Rule)**.** *If* $f(x) = \frac{u(x)}{v(x)}$ *where* $v(x) \neq 0$*, then*

$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}.$$

Proof. Fix $x = a$. Then,

$$\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\
&= \lim_{h\to 0} \frac{\frac{u(a+h)}{v(a+h)} - \frac{u(a)}{v(a)}}{h} \\
&= \lim_{h\to 0} \frac{u(a+h)v(a) - u(a)v(a+h)}{h \cdot v(a)v(a+h)} \\
&= \lim_{h\to 0} \frac{u(a+h)v(a) - u(a)v(a) + u(a)v(a) - u(a)v(a+h)}{h \cdot v(a)v(a+h)} \\
&= \lim_{h\to 0} \frac{v(a)}{v(a)v(a+h)} \frac{u(a+h) - u(a)}{h} + \lim_{h\to 0} \frac{u(a)}{v(a)v(a+h)} \frac{v(a) - v(a+h)}{h} \\
&= \frac{v(a)}{v(a)^2}u'(a) - \frac{u(a)}{v(a)^2}v'(a) \\
&= \frac{u'(a)v(a) - u(a)v'(a)}{v(a)^2}.
\end{aligned}$$

□

Example 4.1. *Differentiate* $f(x) = \dfrac{x^2+1}{x}$.

Let $u(x) = x^2 + 1$ and $v(x) = x$. Then

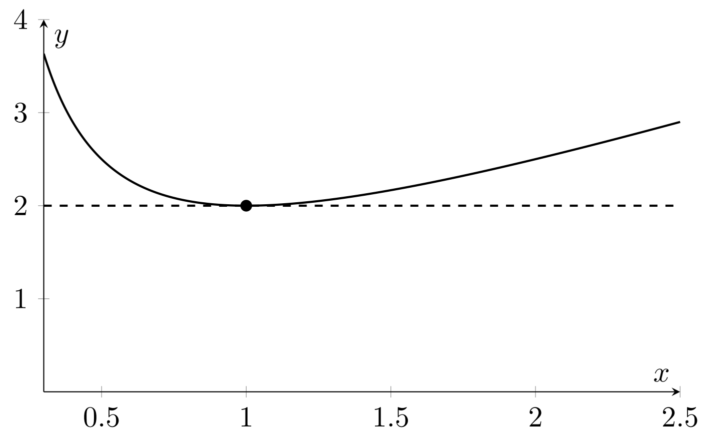
$$f'(x) = \frac{(2x)(x) - (x^2+1)(1)}{x^2} = \frac{x^2-1}{x^2}.$$

5 A Tangent Line Example

Example 5.1. *Find the equation of the tangent line to* $y = \dfrac{x^2+1}{x}$ *at* $x = 1$.

From above, $y' = \dfrac{x^2-1}{x^2}$. At $x = 1$, the slope is 0. Since $y(1) = 2$, the tangent line is

$$y = 2.$$



6 Common Pitfalls

Remark 6.1. *The derivative of a product is* not *the product of the derivatives:*

$$(uv)' \neq u'v'.$$

Remark 6.2. *The derivative of a quotient is* not *the quotient of the derivatives:*

$$(u/v)' \neq u'/v'.$$

7 Why This Matters for Calculus

- Many real-world models involve products and ratios of functions.
- The product and quotient rules are essential for rational functions.
- These rules prepare us for implicit differentiation and related rates.