Polynomial and Rational Functions

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Key Topics

- Polynomial functions and their graphs
- Solving polynomial equations
- Rational functions
- Domain and range of rational functions

1 Polynomial Functions

Definition. *A* polynomial function *is a function of the form*

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_0, a_1, \ldots, a_n$ *are real numbers and* n *is a nonnegative integer.*

Polynomial functions are defined for all real numbers and are continuous everywhere. These properties will be essential when evaluating limits and computing derivatives.

1.1 Graphs of Polynomial Functions

Figure 1 shows a typical example of a polynomial graph.

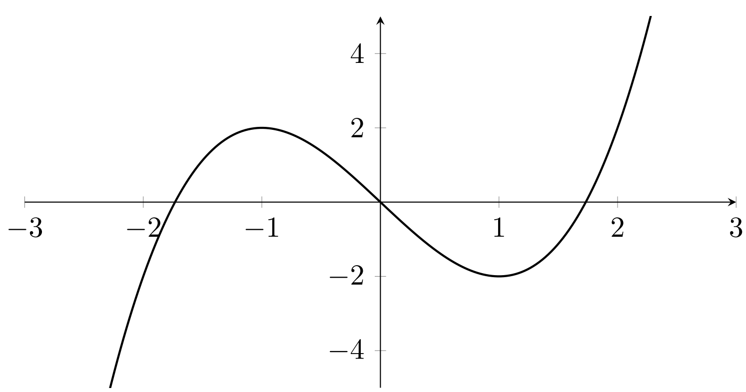

Figure 1: The graph of the polynomial function $f(x) = x^3 - 3x$, illustrating smoothness and end behavior.

Polynomial graphs are smooth and have no breaks or corners, a fact that will be used later when discussing continuity and differentiability.

2 Solving Polynomial Equations

Many problems in calculus reduce to solving polynomial equations, particularly when finding critical points or evaluating limits algebraically.

2.1 Factoring

Example

Solve

$$x^3 - 3x = 0.$$

Factoring gives

$$x(x^2 - 3) = 0,$$

so the solutions are $x = 0$ and $x = \pm\sqrt{3}$.

2.2 Completing the Square

Completing the square is useful when factoring is not straightforward.

Example

Solve

$$x^2 - 4x + 1 = 0.$$

Rewrite as

$$x^2 - 4x = -1,$$

and complete the square:

$$x^2 - 4x + 4 = 3 \quad \Rightarrow \quad (x - 2)^2 = 3.$$

Thus,

$$x = 2 \pm \sqrt{3}.$$

3 Rational Functions

Definition. *A* rational function *is a function of the form*

$$f(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$.

Unlike polynomials, rational functions are not defined for all real numbers. Determining their domains requires algebraic analysis.

4 Domain of a Rational Function

The domain of a rational function consists of all real numbers for which the denominator is nonzero.

Example

Find the domain of

$$f(x) = \frac{x^2 - 1}{x - 1}.$$

Factoring the numerator gives

$$f(x) = \frac{(x-1)(x+1)}{x-1}.$$

The denominator is zero when $x = 1$, so the domain is

$$(-\infty, 1) \cup (1, \infty).$$

Example

Coonsider the following proper rational function

$$f(x) = \frac{x}{x^2 - 1} = \frac{x}{(x-1)(x+1)}.$$

An important algebraic technique known as partial fraction decomposition is used to decompose the rational function into a sum of rationals with simplified denominators. In this example, we have

$$\frac{x}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}.$$

Multiplying both sides by the denominator gives

$$x = A(x+1) + B(x-1),$$

which implies that $A = 1/2$ and $B = -1/2$. Therefore,

$$f(x) = \frac{x}{(x-1)(x+1)} = \frac{1}{2}\frac{1}{x-1} - \frac{1}{2}\frac{1}{x+1}.$$

5 Graphical Behavior of Rational Functions

Although the factor $(x - 1)$ cancels algebraically, the function is still undefined at $x = 1$.

Figure 2 illustrates this behavior.

This type of discontinuity plays a key role in limit computations.

6 Why This Matters for Calculus

- Factoring and algebraic simplification are essential for evaluating limits.
- Rational functions often produce indeterminate forms such as $\frac{0}{0}$.
- Understanding domains prevents illegal algebraic operations.

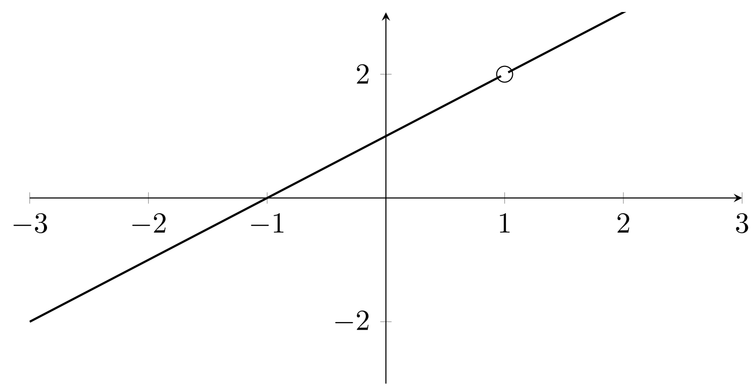

Figure 2: The graph of $f(x) = \dfrac{x^2 - 1}{x - 1}$, showing a hole where the function is undefined.