

# Optimization

Math 140: Calculus with Analytic Geometry

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## 1 Introduction

Many real-world problems involve finding the largest or smallest possible value of a quantity. These are called *optimization problems*. Examples include

- maximizing area or volume,
- minimizing cost or material,
- minimizing travel distance,
- designing efficient containers.

### Strategy for Optimization Problems

Most optimization problems follow the same steps.

1. Draw a diagram and define variables.
2. Write model that clearly identifies the objective and constraints
3. Use constraints to express the objective as a function (with domain) of a single variable
4. Apply first or second derivative test to identify the absolute extrema of the function over the given domain.
5. Interpret the solution in the context of the original problem.

## 2 Examples

### Example 1: Rectangle with fixed perimeter

A rectangle has perimeter 20. What dimensions maximize its area? What is the maximum area.

Let the sides be  $x$  and  $y$ . Then, the problem can be modeled as follows:

$$\begin{aligned} &\text{maximize} && A = xy \\ &\text{subject to} && 2x + 2y = 20, \\ &&& x, y \geq 0 \end{aligned}$$

Therefore, the function we want to maximize is

$$A(x) = x(10 - x) = 10x - x^2,$$

with domain  $0 \leq x \leq 10$ . The first derivative is

$$A'(x) = 10 - 2x.$$

So, we have a critical number at  $x = 5$ . Applying the first derivative test shows that  $x = 5$  is a local max. Also, note that  $A(0) = A(10) = 0$ , so  $x = 5$  is an absolute max over  $[0, 10]$ .

Note that when  $x = 5$ , we have  $y = 5$  and  $A = 25$ .

### Example 2: Rectangle Under a Curve

Find the rectangle of maximum area that can be inscribed under

$$y = 4 - x^2$$

above the  $x$ -axis. What are the dimensions and area of this rectangle?

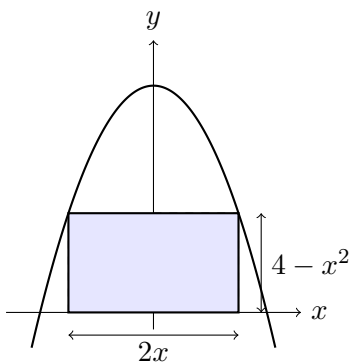


Figure 1: Illustration of problem in Example 2.

This problem can be modeled as follows

$$\begin{aligned} \text{maximize} \quad & A = 2xy \\ \text{subject to} \quad & y = 4 - x^2, \\ & x, y \geq 0 \end{aligned}$$

Therefore, the function we want to maximize is

$$A(x) = 2x(4 - x^2) = 8x - 2x^3,$$

with domain  $0 \leq x \leq 2$ .

The first derivative is

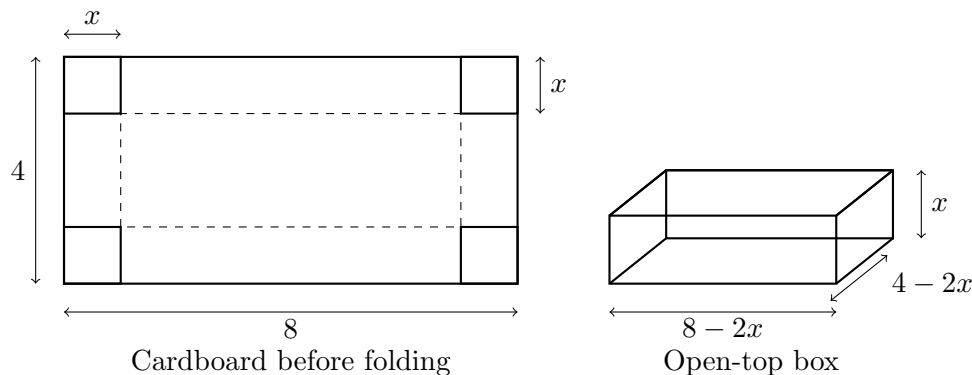
$$A'(x) = 8 - 6x^2.$$

So, the critical number (in our domain) is  $x = \sqrt{\frac{4}{3}}$ . Applying the first derivative test we see that this critical number is a local max. Moreover,  $A(0) = A(2) = 0$ . Hence,  $x = \sqrt{\frac{4}{3}}$  is an absolute max over  $[0, 10]$ .

Note that when  $x = \sqrt{\frac{4}{3}}$ , we have  $y = 8/3$  and  $A = 64/9$ .

### Example 3: Open-Top Box

An  $8 \times 4$  sheet of cardboard is used to construct an open-top box by cutting squares from each corner. Find size of square cutout that maximizes the volume of the open box. What are the dimensions and volume of this box?



This problem can be modeled as follows

$$\begin{aligned} &\text{maximize} && V = x(4 - 2x)8 - 2x \\ &\text{subject to} && x \geq 0, \\ &&& x \leq 2 \end{aligned}$$

Therefore, the function we want to maximize is

$$V(x) = x(4 - 2x)(8 - 2x) = 4x^3 - 24x^2 + 32x,$$

with domain  $0 \leq x \leq 2$ .

The first derivative is

$$\begin{aligned} V'(x) &= 12x^2 - 48x + 32 \\ &= 12(x^2 - 4x + 8/3) \\ &= 12((x - 2)^2 + 8/3 - 4) \\ &= 12((x - 2)^2 - 4/3). \end{aligned}$$

So, the critical number in our domain is  $x = 2 - 2/\sqrt{3}$ . Applying the first derivative test we see that this critical number is a local max. Since  $V(0) = V(2) = 0$ , it follows that this critical number is an absolute max over the interval  $[0, 2]$ .

Note that when  $x = 2 - 2/\sqrt{3}$ , we have a volume of

$$V = 4 \left(2 - \frac{2}{\sqrt{3}}\right)^3 - 24 \left(2 - \frac{2}{\sqrt{3}}\right)^2 + 32 \left(2 - \frac{2}{\sqrt{3}}\right) \approx 12.32.$$

### Example 4: Minimum Distance Problem

Find the point on the parabola  $y = x^2$  that is closest to the point  $(0, 1)$ .

Note that the distance between the point  $(0, 1)$  and the point  $(x, y)$  is given by Pythagorean's Theorem:

$$D = \sqrt{(x - 0)^2 + (y - 1)^2}.$$

Since the square root function is strictly increasing, we can find the point that minimizes the distance by considering the distance squared. Hence, the problem can be modeled as follows

$$\begin{aligned} &\text{maximize} && f = x^2 + (y - 1)^2 \\ &\text{subject to} && y = x^2, \\ &&& x \in \mathbb{R} \end{aligned}$$

Therefore, the function we want to maximize is

$$f(x) = x^2 + (x^2 - 1)^2,$$

with domain  $-\infty < x < \infty$ .

The first derivative is given by

$$\begin{aligned} f'(x) &= 2x + 2(x^2 - 1)(2x) \\ &= 4x^3 - 2x \\ &= 2x(2x^2 - 1) \end{aligned}$$

So, the critical numbers are given by  $x = 0, \pm \frac{1}{\sqrt{2}}$ . Applying the first derivative test we find that  $x = 0$  is a local max and  $x = \pm \frac{1}{\sqrt{2}}$  are local min. Furthermore, since  $\lim_{x \rightarrow \pm\infty} f(x) = \pm\infty$ , it follows that  $x = \pm \frac{1}{\sqrt{2}}$  are absolute min.

Note that when  $x = \pm \frac{1}{\sqrt{2}}, y = \frac{1}{2}$  and the distance from  $(0, 1)$  to  $(x, y)$  is given by

$$D = \sqrt{x^2 + (y - 1)^2} = \sqrt{\frac{1}{2} + \frac{1}{4}} = \frac{\sqrt{3}}{2}.$$

### Example 5: Cylinder Inside a Cone

A right circular cone has height 10 and radius 6. Find the dimensions of the right circular cylinder of maximum volume that can be inscribed in the cone.

Let  $r$  and  $h$  denote the radius and height of the cylinder, respectively. Using similar triangles, we find that

$$\frac{10 - h}{r} = \frac{10}{6} = \frac{5}{3},$$

that is,  $h = 10 - \frac{5}{3}r$ . Therefore, this problem can be modeled as follows

$$\begin{aligned} &\text{maximize} && V = \pi r^2 h \\ &\text{subject to} && h + \frac{5}{3}r = 10, \\ &&& h, r \geq 0 \end{aligned}$$

Furthermore, the function we want to maximize is

$$V(r) = \pi r^2 \left(10 - \frac{5}{3}r\right) = \pi \left(10r^2 - \frac{5}{3}r^3\right),$$

with domain  $0 \leq r \leq 6$ . The first derivative is given by

$$V'(r) = \pi (20r - 5r^2) = 5\pi r (4 - r).$$

Hence, the critical numbers occur at  $r = 0$  and  $r = 4$ .

Note that  $V(0) = v(6) = 0$  and  $V(4) = \frac{160}{3}\pi$ . Therefore, the maximum volume the cylinder occurs when  $r = 4$ , which corresponds to  $h = 10/3$ .

### Example 6: Cylindrical Can

A cylindrical can must hold  $1000 \text{ cm}^3$  of liquid. Find the dimensions that minimize the surface area.

This problem can be modeled as follows

$$\begin{aligned} &\text{maximize} && S = 2\pi r^2 + 2\pi r h \\ &\text{subject to} && \pi r^2 h = 1000, \\ &&& r, h \geq 0 \end{aligned}$$

Therefore, the function we want to maximize is

$$S(r) = 2\pi r^2 + \frac{2000}{r},$$

with domain  $r > 0$ .

The first derivative of this function is given by

$$S'(r) = 4\pi r - \frac{2000}{r^2}.$$

Hence, the critical numbers are given by  $r = 0$  and

$$r = \left(\frac{500}{\pi}\right)^{1/3} \approx 5.42.$$

Applying the first derivative test, we find the later critical number is a local min. Since  $\lim_{r \rightarrow 0^+} S(r) = +\infty$  and  $\lim_{r \rightarrow +\infty} S(r) = +\infty$ , it follows that this critical number is an absolute max.

Note that when  $r = \left(\frac{500}{\pi}\right)^{1/3}$ , we have  $h = 2\left(\frac{500}{\pi}\right)^{1/3} = 2r$  and  $S \approx 553.58$ .

## 3 Summary

Optimization problems involve

- translating a real-world situation into mathematics,
- expressing the quantity to be optimized as a function,
- using derivatives to locate extrema.

These problems illustrate one of the most important applications of calculus.