Limits of Transcendental Functions

Math 140: Calculus with Analytic Geometry

Key Topics

- Limits of trigonometric functions at points in their domains
- The squeeze theorem
- Special trigonometric limits
- Limits of exponential and logarithmic functions
- Asymptotic behavior of exponentials and logarithms
- Combining these facts with algebraic limit laws

1 Limits of Trigonometric Functions

In this section we record basic limit facts for $\sin(x)$, $\cos(x)$, and $\tan(x)$. Although these facts can be justified using continuity, we will treat them as known limit rules.

Theorem 1. *Let c be a real number.*

1. $\lim_{x \to c} \sin(x) = \sin(c)$.
2. $\lim_{x \to c} \cos(x) = \cos(c)$.
3. $\lim_{x \to c} \tan(x) = \tan(c)$, *provided that c is in the domain of* $\tan(x)$.

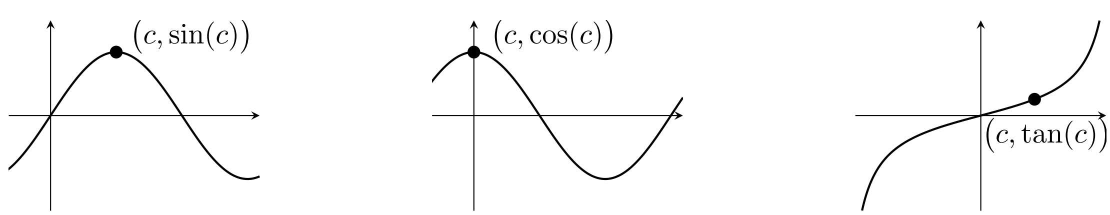

Figure 1: The values of $\sin(x)$, $\cos(x)$, and $\tan(x)$ approach their corresponding function values as x approaches c (when c is in the domain).

Figure 1 illustrates the idea that near a point c in the domain, the function values of $\sin(x)$, $\cos(x)$, and $\tan(x)$ approach $\sin(c)$, $\cos(c)$, and $\tan(c)$, respectively.

2 The Squeeze Theorem

Theorem 2 (Squeeze Theorem)**.** *Suppose there exists an interval around c (excluding c) on which*

$$g(x) \leq f(x) \leq h(x).$$

If

$$\lim_{x \to c} g(x) = L \quad \text{and} \quad \lim_{x \to c} h(x) = L,$$

then

$$\lim_{x \to c} f(x) = L.$$

3 Special Trigonometric Limits

Theorem 3 (Special Limits)**.**

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1, \qquad \lim_{x \to 0} \frac{1 - \cos(x)}{x} = 0.$$

The key geometric comparison (illustrated below) shows that for $0 < x < \frac{\pi}{2}$,

$$\sin(x) \leq x \leq \tan(x).$$

Dividing by $x > 0$ gives

$$\cos(x) \leq \frac{\sin(x)}{x} \leq 1,$$

and applying the squeeze theorem as $x \to 0$ yields $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$.

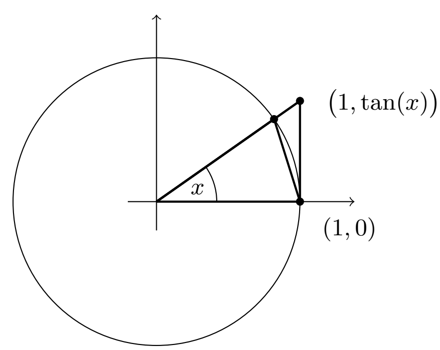

Figure 2: For $0 < x < \frac{\pi}{2}$, the geometry gives $\frac{\tan(x)}{2} \geq \frac{x}{2} \geq \frac{\sin(x)}{2}$.

Figure 2 is used with the squeeze theorem to establish

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$$

Next, we will establish that $\lim_{x \to 0} \frac{1 - \cos(x)}{x} = 0$. We start with the identity

$$1 - \cos(x) = \frac{(1 - \cos(x))(1 + \cos(x))}{1 + \cos(x)} = \frac{1 - \cos^2(x)}{1 + \cos(x)} = \frac{\sin^2(x)}{1 + \cos(x)}.$$

Then, for $x \neq 0$,

$$\frac{1-\cos(x)}{x} = \left(\frac{\sin(x)}{x}\right)\left(\frac{\sin(x)}{1+\cos(x)}\right).$$

As $x \to 0$,

$$\frac{\sin(x)}{x} \to 1, \qquad \sin(x) \to 0, \qquad 1+\cos(x) \to 2,$$

so

$$\frac{\sin(x)}{1+\cos(x)} \to \frac{0}{2} = 0.$$

By the product law for limits,

$$\lim_{x\to 0} \frac{1-\cos(x)}{x} = \left(\lim_{x\to 0} \frac{\sin(x)}{x}\right)\left(\lim_{x\to 0} \frac{\sin(x)}{1+\cos(x)}\right) = 1 \cdot 0 = 0.$$

4 Limits of Exponential and Logarithmic Functions

We now state basic limit facts for exponentials and logarithms. These will also be treated as known rules.

Theorem 4. *Let $b > 0$, $b \neq 1$.*

1. $\lim_{x\to c} b^x = b^c$ *for every real number c.*
2. $\lim_{x\to c} \log_b(x) = \log_b(c)$ *provided that $c > 0$.*

Theorem 5 (Asymptotic Limits)**.** *Let $b > 0$, $b \neq 1$.*

1. $\lim_{x\to\infty} b^x = \begin{cases} 0, & 0 < b < 1, \\ \infty, & b > 1, \end{cases}$
2. $\lim_{x\to 0^+} \log_b(x) = -\infty,$
3. $\lim_{x\to\infty} \log_b(x) = +\infty.$

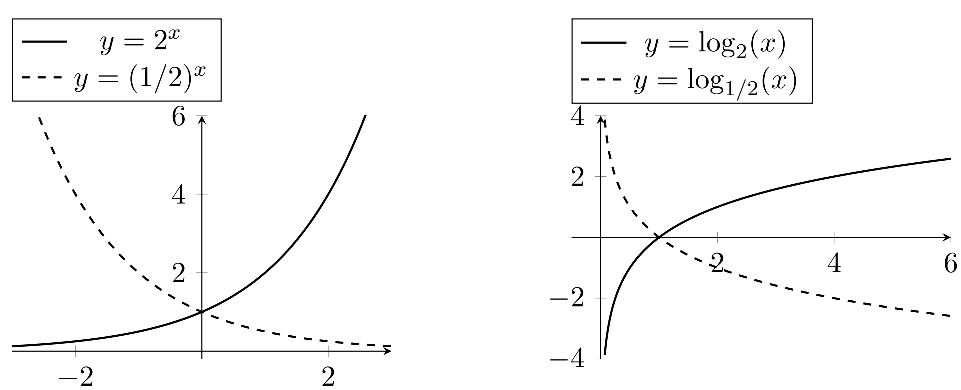

Figure 3: Exponential and logarithmic behavior: $2^x \to \infty$ and $(1/2)^x \to 0$ as $x \to \infty$, while $\log_2(x) \to -\infty$ as $x \to 0^+$ and $\log_2(x) \to \infty$ as $x \to \infty$.

Figure 3 illustrates the limiting behavior of exponentials and logarithms, including their asymptotes.

5 Examples Combining Limit Laws with Transcendental Limits

Throughout these examples we use the algebraic limit laws (sum, product, quotient).

Example 1

Evaluate

$$\lim_{x\to\pi/3} \left(3\sin(x) - 2\cos(x)\right).$$

Using the trigonometric limit facts,

$$\lim_{x\to\pi/3} 3\sin(x) = 3\sin\left(\frac{\pi}{3}\right), \qquad \lim_{x\to\pi/3} 2\cos(x) = 2\cos\left(\frac{\pi}{3}\right).$$

Therefore,

$$\lim_{x\to\pi/3} \left(3\sin(x) - 2\cos(x)\right) = 3\sin\left(\frac{\pi}{3}\right) - 2\cos\left(\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2} - 1.$$

Example 2

Evaluate

$$\lim_{x\to 0} \frac{\sin(5x)}{x}.$$

Rewrite as

$$\frac{\sin(5x)}{x} = 5 \cdot \frac{\sin(5x)}{5x}.$$

Let $u = 5x$. Then as $x \to 0$, we have $u \to 0$, and therefore

$$\lim_{x\to 0} 5 \cdot \frac{\sin(5x)}{5x} = 5 \cdot \lim_{u\to 0} \frac{\sin(u)}{u} = 5.$$

Example 3

Evaluate

$$\lim_{x\to 0} \frac{\sin(x)}{2x}.$$

Rewrite as

$$\frac{\sin(x)}{2x} = \frac{1}{2} \cdot \frac{\sin(x)}{x}.$$

Using the special limit,

$$\lim_{x\to 0} \frac{\sin(x)}{2x} = \frac{1}{2}.$$

Example 4

Evaluate

$$\lim_{x\to 2} \left(\frac{2^x + 1}{x}\right).$$

Using the quotient law and the exponential limit fact,

$$\lim_{x\to 2} \left(\frac{2^x + 1}{x}\right) = \frac{\lim_{x\to 2}(2^x + 1)}{\lim_{x\to 2} x} = \frac{2^2 + 1}{2} = \frac{5}{2}.$$

Example 5

Evaluate

$$\lim_{x\to 1} \left(\log_2(x) + x\cos(x)\right).$$

We use the sum law and the basic pointwise limits:

$$\lim_{x\to 1} \log_2(x) = \log_2(1) = 0, \qquad \lim_{x\to 1} x\cos(x) = \left(\lim_{x\to 1} x\right)\left(\lim_{x\to 1} \cos(x)\right) = 1 \cdot \cos(1).$$

Therefore,

$$\lim_{x\to 1} \left(\log_2(x) + x\cos(x)\right) = \cos(1).$$

6 Why This Matters for Calculus

- The limit $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$ is the key ingredient in deriving derivative rules for $\sin(x)$ and $\cos(x)$.
- Exponential and logarithmic limits provide rules for evaluating many applied limits.
- Combining transcendental limit rules with algebraic limit laws allows rapid evaluation of many limits.