Finding Limits Algebraically

Math 140: Calculus with Analytic Geometry

Key Topics

- Basic limits and one-sided infinite behavior
- Algebraic properties (limit laws)
- Limits of polynomials and rational functions
- Rational functions with holes
- Piecewise limits built from polynomial and rational pieces

1 Basic Limits

We begin with a short list of basic limits. These are the building blocks for the algebraic techniques developed next.

Theorem 1 (Basic Limits)**.** *Let c be a real number.*

1. $\lim_{x \to c} k = k$ *for any constant* k.
2. $\lim_{x \to c} x = c$.
3. $\lim_{x \to 0^-} \frac{1}{x} = -\infty$.
4. $\lim_{x \to 0^+} \frac{1}{x} = +\infty$.

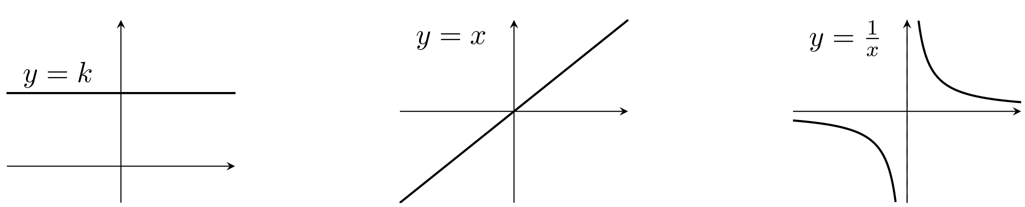

Figure 1: Graphs for the basic limits: a constant (left), the identity (middle), and $y = \frac{1}{x}$ (right).

Figure 1 visually supports the basic limits above. In particular, the graph of $y = \frac{1}{x}$ shows that as $x \to 0^+$ the function increases without bound, while as $x \to 0^-$ the function decreases without bound.

2 Algebraic Properties of Limits

The following limit laws allow us to evaluate many limits by reducing them to the basic limits. Note that these laws can all be justified using the ϵ-δ definition of the limit.

Theorem 2 (Limit Laws). *Assume that* $\lim_{x\to c} f(x)$ *and* $\lim_{x\to c} g(x)$ *exist.*

1. *(Sum)* $\lim\limits_{x\to c} \big(f(x) + g(x)\big) = \lim\limits_{x\to c} f(x) + \lim\limits_{x\to c} g(x)$.

2. *(Product)* $\lim\limits_{x\to c} \big(f(x)g(x)\big) = \Big(\lim\limits_{x\to c} f(x)\Big)\Big(\lim\limits_{x\to c} g(x)\Big)$.

3. *(Quotient)* $\lim\limits_{x\to c} \left(\dfrac{f(x)}{g(x)}\right) = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ *provided* $\lim_{x\to c} g(x) \neq 0$.

Example

We will demonstrate the Limit sum law using the ϵ-δ definition of the limit. Suppose that $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = L'$. Let $\epsilon > 0$. Then, there exists a $\delta > 0$ such that

$$0 < |x - c| < \delta \implies |f(x) - L| < \frac{\epsilon}{2}.$$

Similarly, there exists a $\delta' > 0$ such that

$$0 < |x - c| < \delta' \implies |g(x) - L'| < \frac{\epsilon}{2}$$

If $0 < |x - c| < \min\{\delta, \delta'\}$, then both of the above implications hold. Therefore, we have

$$|(f(x) + g(x)) - (L + L')| = |(f(x) - L) + (g(x) - L')| \leq |f(x) - L| + |g(x) - L'| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

So, by definition, we have

$$\lim_{x\to c}(f(x) + g(x)) = L + L' = \lim_{x\to c} f(x) + \lim_{x\to c} g(x).$$

Example

Evaluate

$$\lim_{x\to 3}\left(2x^2 - \frac{1}{x}\right).$$

We use the sum law, product law, and quotient law:

$$\lim_{x\to 3}(2x^2) = 2\left(\lim_{x\to 3} x\right)\left(\lim_{x\to 3} x\right) = 2\cdot 3\cdot 3 = 18, \qquad \lim_{x\to 3}\frac{1}{x} = \frac{\lim_{x\to 3} 1}{\lim_{x\to 3} x} = \frac{1}{3}.$$

Therefore,

$$\lim_{x\to 3}\left(2x^2 - \frac{1}{x}\right) = 18 - \frac{1}{3} = \frac{53}{3}.$$

Figure 2 illustrates the limit computed using the limit laws.

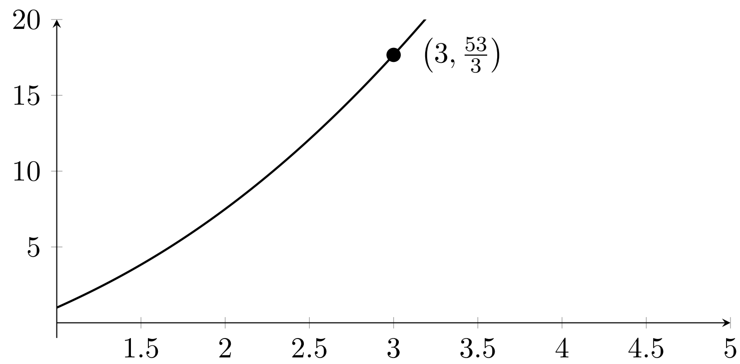

Figure 2: The function values of $y = 2x^2 - \frac{1}{x}$ approach $\frac{53}{3}$ as $x \to 3$.

3 Limits of Polynomials and Rational Functions

Once we know how limits distribute across sums, products, and quotients, we can evaluate limits of many algebraic functions.

Theorem 3. *If $p(x)$ is a polynomial, then for every real number c,*

$$\lim_{x \to c} p(x) = p(c).$$

Remark. *This follows by writing a polynomial as a sum of constant multiples of products of x with itself, and then applying the limit laws repeatedly.*

If $r(x) = \dfrac{p(x)}{q(x)}$ is a rational function and $q(c) \neq 0$, then

$$\lim_{x \to c} r(x) = \frac{p(c)}{q(c)}.$$

When $q(c) = 0$, direct substitution fails and additional algebra is required.

4 Rational Functions with Holes

In this section we focus on rational functions that simplify after factoring, producing a hole in the graph.

Example 1

Evaluate

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}.$$

Direct substitution gives $\frac{0}{0}$. Factor:

$$\frac{x^2 - 1}{x - 1} = \frac{(x-1)(x+1)}{x - 1}.$$

For $x \neq 1$, this simplifies to $x + 1$, so

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} (x + 1) = 2.$$

Figure 3 shows why the limit exists even though the function is undefined at $x = 1$.

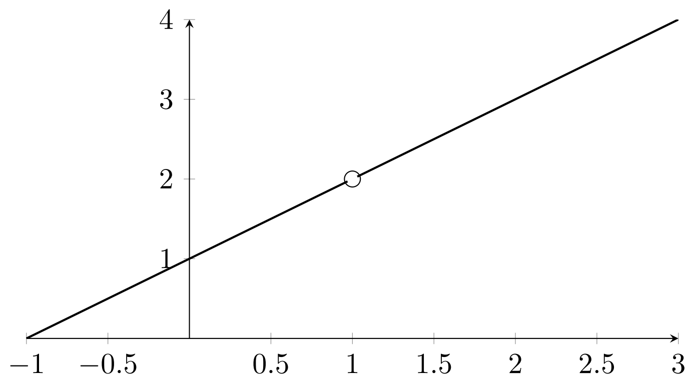

Figure 3: Near $x = 1$, $\dfrac{x^2 - 1}{x - 1}$ agrees with $y = x + 1$ except for a hole at $(1, 2)$.

Example 2

Evaluate

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}.$$

By substitution we obtain $\frac{0}{0}$. Factor:

$$\frac{x^2 - 4}{x - 2} = \frac{(x - 2)(x + 2)}{x - 2}.$$

For $x \neq 2$, this simplifies to $x + 2$, so

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} (x + 2) = 4.$$

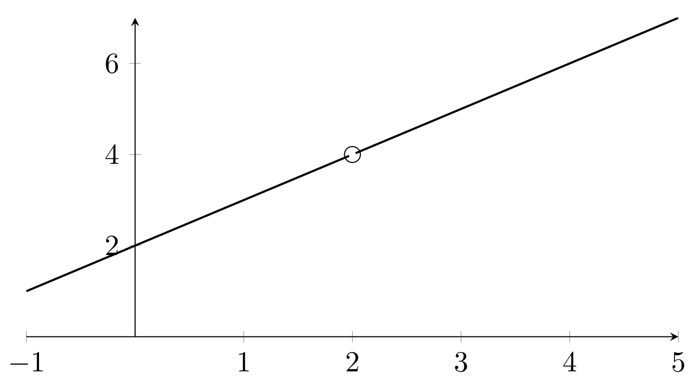

Figure 4: Near $x = 2$, $\dfrac{x^2 - 4}{x - 2}$ agrees with $y = x + 2$ except for a hole at $(2, 4)$.

Figure 4 illustrates a second limit evaluated by factoring.

5 A Piecewise Limit with Polynomial and Rational Parts

Define

$$f(x) = \begin{cases} x^2 - 1, & x < 1, \\ \dfrac{x^2 - 1}{x - 1}, & x > 1, \\ 0, & x = 1. \end{cases}$$

We evaluate $\lim_{x \to 1} f(x)$ by comparing the one-sided limits.

For $x < 1$, we have $f(x) = x^2 - 1$, so

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2 - 1) = 1^2 - 1 = 0.$$

For $x > 1$, we simplify the rational piece:

$$\frac{x^2 - 1}{x - 1} = x + 1 \quad (x \neq 1),$$

so

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x + 1) = 2.$$

Since the one-sided limits are not equal, the two-sided limit does not exist:

$$\lim_{x \to 1} f(x) \text{ does not exist.}$$

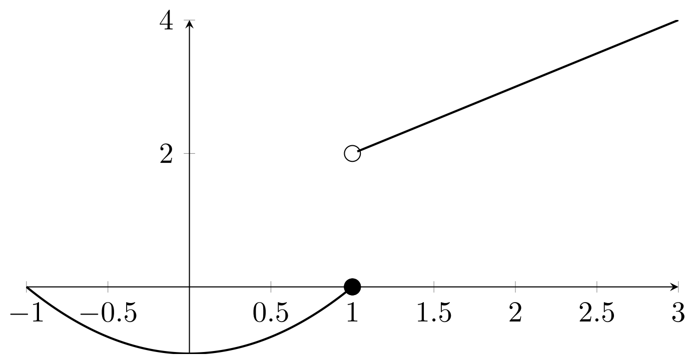

Figure 5: A piecewise function with a polynomial part and a rational part. The one-sided limits at $x = 1$ disagree.

Figure 5 shows the left-hand approach toward 0 and the right-hand approach toward 2.

6 Why This Matters for Calculus

- The limit laws let us evaluate limits of many algebraic expressions by reducing them to basic ones.
- Limits of polynomials and rational functions are the foundation for continuity and derivative rules.
- Piecewise functions force us to compare one-sided behavior, which is essential in later topics.