Inverse Functions

Math 140: Calculus with Analytic Geometry

Key Topics

- Function notation, domain, and range
- Function composition
- One-to-one functions
- Inverse functions and reflection across $y = x$
- Restricting domains to obtain inverses

1 Functions, Domain, and Range

Definition. *A function $f\colon A \to B$ assigns to each element of the set A exactly one element of the set B. The set A is called the* domain *of f, and the set B is called the* codomain. *The* range *of f, denoted* Range(f), *is the set of all $y \in B$ such that $f(x) = y$ for some $x \in A$.*

2 Function Composition

Definition. *If $f\colon A \to B$ and $g\colon B \to C$, the* composition *of g with f is the function*

$$(g \circ f)\colon A \to C, \qquad (g \circ f)(x) = g(f(x)).$$

Example

Let $f(x) = 2x + 1$ and $g(x) = x^2$. Then

$$(g \circ f)(x) = (2x + 1)^2, \qquad (f \circ g)(x) = 2x^2 + 1.$$

Remark. *In general, $g \circ f \neq f \circ g$.*

3 One-to-One Functions

Definition. *A function $f\colon A \to B$ is* one-to-one *if*

$$f(x_1) = f(x_2) \implies x_1 = x_2.$$

Note that $\implies$ denotes the logical implication, the statement $p \implies q$ can be read as "if p then q" or "q whenever p."

Graphically, a function is one-to-one if every horizontal line intersects its graph at most once.

4 Inverse Functions

Definition. *Let* $f\colon A \to B$ *be a function. The* inverse function *of* f *is a function*

$$f^{-1}\colon \text{Range}(f) \to A$$

such that

$$f^{-1}(f(x)) = x \quad \textit{for all } x \in A, \qquad f(f^{-1}(y)) = y \quad \textit{for all } y \in \text{Range}(f).$$

Remark. *An inverse function exists if and only if* f *is one-to-one. When passing to the inverse, the domain and range are interchanged.*

Geometric interpretation

The graph of f^{-1} is a reflection of the graph of f across the line $y = x$.

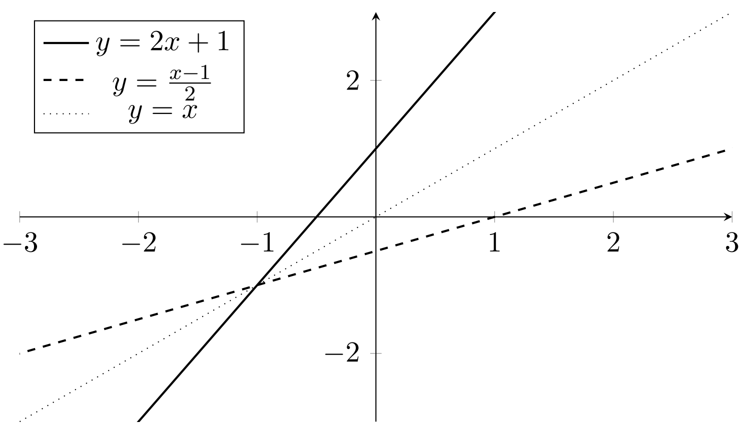

Figure 1: The graphs of a function and its inverse, reflected across the line $y = x$.

Figure 1 illustrates that each point (a, b) on the graph of f corresponds to the point (b, a) on the graph of f^{-1}.

5 Functions Without Inverses

Many commonly encountered functions are not one-to-one on their natural domains.

Quadratic function. The function $f(x) = x^2$ with domain $\mathbb{R}$ is not one-to-one, since $f(2) = f(-2)$.

Figure 2 shows that horizontal lines intersect the graph of x^2 more than once.

Restricting the domain. If we restrict the domain to $[0, \infty)$, then $f\colon [0, \infty) \to [0, \infty)$ becomes one-to-one and has inverse $f^{-1}(x) = \sqrt{x}$.

Trigonometric examples. The cosine function is not one-to-one on $\mathbb{R}$, but restricting its domain to $[0, \pi]$ produces an inverse. The tangent function is one-to-one on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and has an inverse on that interval.

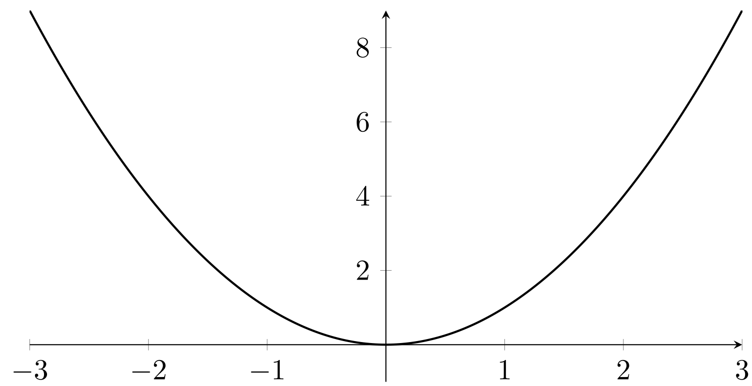

Figure 2: The graph of $f(x) = x^2$, which fails the horizontal line test.

6 Finding Inverses Algebraically

Example (Linear)

Let $f(x) = 3x - 5$ with domain $\mathbb{R}$ and range $\mathbb{R}$. Solving $y = 3x - 5$ for x gives

$$f^{-1}(x) = \frac{x+5}{3}.$$

Example (Rational)

Let $f(x) = \dfrac{x-1}{x+2}$ with domain $\mathbb{R} \setminus \{-2\}$ and range $\mathbb{R} \setminus \{1\}$. Solving $y = \dfrac{x-1}{x+2}$ for x yields

$$f^{-1}(x) = \frac{-1-2x}{x-1},$$

with domain $\mathbb{R} \setminus \{1\}$ and range $\mathbb{R} \setminus \{-2\}$.

7 Why This Matters for Calculus

- Inverse functions explain the meaning of inverse trigonometric and logarithmic functions.
- Domain restrictions determine where derivatives and integrals are defined.
- Inverses naturally arise when solving equations involving derivatives.