The Intermediate Value Theorem and Applications

Math 140: Calculus with Analytic Geometry

Key Topics

- Statement and interpretation of the Intermediate Value Theorem (IVT)
- Using the IVT to prove existence of solutions to equations
- Motivating the bisection method for approximating roots

1 The Intermediate Value Theorem

Theorem 1 (Intermediate Value Theorem)**.** *Let f be continuous on the interval $[a, b]$, and let N be any number between $f(a)$ and $f(b)$. Then there exists at least one number c in (a, b) such that*

$$f(c) = N.$$

Intuitively, a continuous function cannot jump from $f(a)$ to $f(b)$ without taking on every intermediate value along the way.

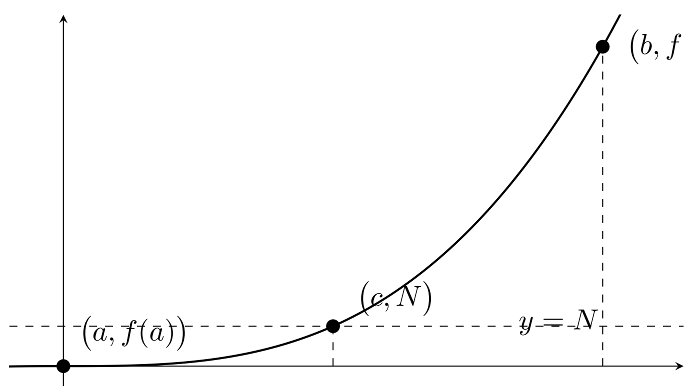

Figure 1: An illustration of the IVT: for a continuous function on $[a, b]$, any value N between $f(a)$ and $f(b)$ occurs at some $c \in (a, b)$.

Figure 1 shows how the horizontal line $y = N$ must intersect the curve at some point $\big(c, N\big)$.

2 Using the IVT to Show an Equation Has a Solution

A common application is to prove that an equation $f(x) = 0$ has a solution by checking that f changes sign.

Example 1

Show that the equation

$$x^3 - 3x + 1 = 0$$

has a solution in the interval $(0, 1)$.

Let $f(x) = x^3 - 3x + 1$. Since f is a polynomial, it is continuous on $[0, 1]$. Compute:

$$f(0) = 1 > 0, \qquad f(1) = 1 - 3 + 1 = -1 < 0.$$

Since 0 lies between $f(0)$ and $f(1)$, the IVT guarantees a number $c \in (0, 1)$ such that $f(c) = 0$.

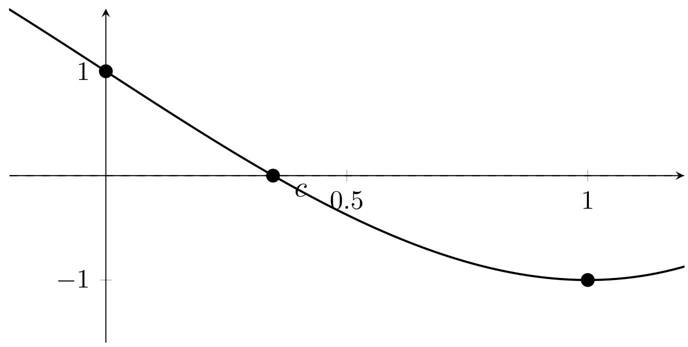

Figure 2: Since $f(0) > 0$ and $f(1) < 0$, the IVT guarantees an x-intercept for $f(x) = x^3 - 3x + 1$ in $(0, 1)$.

In Figure 2, the point labeled c marks an approximate location of the root guaranteed by the IVT.

3 The Bisection Method

The bisection method turns the IVT sign-change idea into a reliable root-approximation algorithm.

Definition (Bisection Method)**.** *Let f be continuous on $[a_0, b_0]$ and suppose $f(a_0)$ and $f(b_0)$ have opposite signs. Define $m_n = \dfrac{a_n + b_n}{2}$. Then:*

- *If $f(m_n) = 0$, then m_n is a root.*
- *If $f(a_n)$ and $f(m_n)$ have opposite signs, set $[a_{n+1}, b_{n+1}] = [a_n, m_n]$.*
- *Otherwise, set $[a_{n+1}, b_{n+1}] = [m_n, b_n]$.*

This produces nested intervals $[a_n, b_n]$ that trap at least one root.

Example 2: Approximating the Root from Example 1

We apply bisection to $f(x) = x^3 - 3x + 1$ on the initial interval $[0, 1]$.

Step table (first few iterations).

n	a_n	b_n	$m_n = \frac{a_n+b_n}{2}$	sign of $f(m_n)$
0	0	1	0.5	$-$
1	0	0.5	0.25	$+$
2	0.25	0.5	0.375	$-$
3	0.25	0.375	0.3125	$+$

So after four rows, the root lies in the interval [0.3125, 0.375].

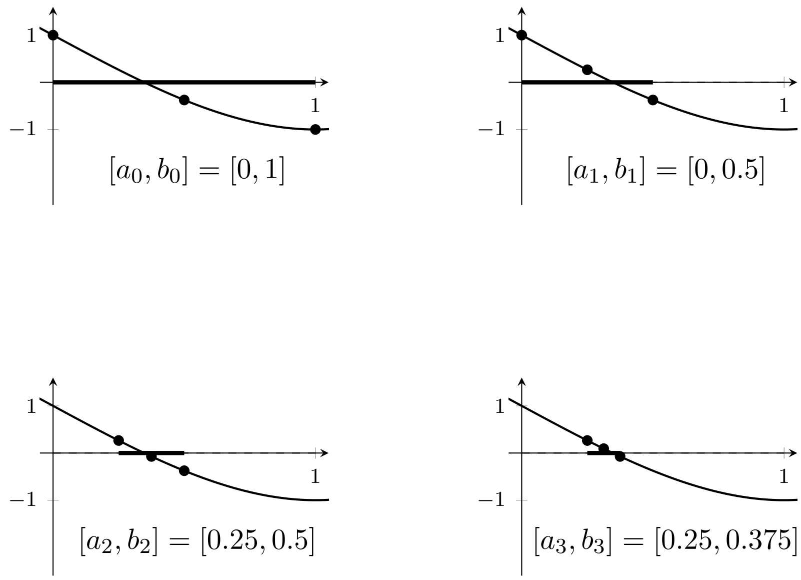

Figure 3: Bisection method illustrated on repeated plots: each step chooses a midpoint and keeps the subinterval where f changes sign. The thick segment on the x-axis shows the current interval $[a_n, b_n]$.

Figure 3 shows how the bisection method produces a sequence of nested intervals trapping a root.

4 Why This Matters for Calculus

- The IVT is a powerful existence theorem for continuous functions.
- It provides a rigorous foundation for root-finding methods.
- The bisection method is reliable: if the hypotheses hold, it will converge to a root.