

Homework 5

Math 140: Calculus with Analytic Geometry

Answer Key

1. Changing volumes.

A cylindrical tank has a fixed radius of 2 m and vertical sides. Water drains at $4\pi \text{ m}^3/\text{min}$ with no inflow. Let h be the water depth and V the volume of water. Assume the surface is horizontal and the tank is not empty.

- (a) Express V in terms of h . State dV/dt , including its sign.
- (b) Find dh/dt . Interpret its sign and include units.
- (c) Compare the water-level rates when $h = 1$ m and when $h = 2$ m. Explain your answer using the tank's shape.

Solution. Here primes denote derivatives with respect to time. (a) $V = \pi(2)^2h = 4\pi h$ and $V' = -4\pi \text{ m}^3/\text{min}$. (b) $V' = 4\pi h'$, so $h' = -1 \text{ m/min}$: the level falls at 1 m/min. (c) The rates are the same, since $h' = -1$ does not depend on h . The horizontal cross-sectional area is constant.

2. Changing distances.

A boat is pulled toward a dock by a rope passing through a fixed ring 6 ft above the boat's attachment point. The rope between the ring and the boat is straight and shortens at 4 ft/s. The boat moves horizontally toward the dock, and its attachment point stays at a constant height. Let x be the horizontal distance from the attachment point to the point directly below the ring, and let s be the rope length between the ring and the boat.

- (a) Write an equation relating x and s . State ds/dt , including its sign.
- (b) Find dx/dt when $x = 8$ ft. How fast is the boat moving toward the dock?
- (c) At this instant, compare the boat's speed with the rope's shortening rate. Explain using your differentiated equation.

Solution. Here primes denote derivatives with respect to time. (a) $x^2 + 36 = s^2$ and $s' = -4 \text{ ft/s}$. (b) $2xx' = 2ss'$. At $x = 8$, $s = 10$, so $x' = (10/8)(-4) = -5 \text{ ft/s}$. The boat moves toward the dock at 5 ft/s. (c) Since $s > x > 0$, $|x'| = (s/x)|s'| > |s'|$; the boat's speed is greater.

3. Changing water levels.

A trough has a constant triangular cross section and is 10 m long. Each end is an isosceles triangle with its vertex at the bottom, height 2 m, and top width 4 m. Water enters at $10 \text{ m}^3/\text{min}$ with no outflow. Let h be the water depth and w the width of the water surface. Assume $0 < h < 2$ and a horizontal water surface.

- (a) Use similar triangles to relate w and h . Express the water volume V in terms of h alone.
- (b) Find the rate at which the water level rises when $h = 1$ m.
- (c) With the same inflow, does the level rise faster at $h = 1/2$ m or at $h = 1$ m? Justify mathematically and explain in terms of the trough's shape.

Solution. Here primes denote derivatives with respect to time. (a) $w/h = 4/2 = 2$, so $w = 2h$ and $V = 10(wh/2) = 10h^2$. (b) $V' = 20hh' = 10$, so $h' = 1/(2h) = 1/2$ m/min at $h = 1$. (c) At $h = 1/2$, $h' = 1$ m/min, twice the rate at $h = 1$. The narrower water surface requires less added volume for the same small rise.

4. Linear approximations and differentials.

Let $f(x) = x^{1/3}$ for $x > 0$.

- (a) Find the linearization $L(x)$ at $a = 8$ and use it to approximate $\sqrt[3]{8.12}$.
- (b) Set $dx = 0.12$ at $x = 8$. Find $dy = f'(8)dx$ and explain how it gives the same estimate as part (a). Write an exact expression for the actual change Δy .
- (c) Use the sign of f'' to decide whether the estimate in (a) is an overestimate or an underestimate, and whether dy is larger or smaller than Δy .

Solution. (a) $f(8) = 2$, $f'(8) = 1/12$, and $L(x) = 2 + (x - 8)/12$, giving $\sqrt[3]{8.12} \approx 2.01$. (b) $dy = (1/12)(0.12) = 0.01$, so $f(8) + dy = 2.01 = L(8.12)$. The actual change is $\Delta y = \sqrt[3]{8.12} - 2$. (c) $f''(x) = -2/(9x^{5/3}) < 0$ for $x > 0$, so the tangent line overestimates the function: $dy > \Delta y$.

5. L'Hôpital's rule.

Evaluate each limit. State the indeterminate form before every application of L'Hôpital's rule and justify its use.

- (a) $\lim_{x \rightarrow 0} \frac{\sin(3x)}{e^{2x} - 1}$.
- (b) $\lim_{x \rightarrow 0} \frac{\ln(1+x) - x}{x^2}$.
- (c) $\lim_{x \rightarrow \infty} \frac{\ln x}{x^2}$.

Solution. The functions are differentiable on the relevant approaching intervals, and the denominators are nonzero there. (a) The form is $0/0$ and $2e^{2x} > 0$:

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{e^{2x} - 1} = \lim_{x \rightarrow 0} \frac{3 \cos(3x)}{2e^{2x}} = \frac{3}{2}.$$

(b) The form is $0/0$ before both applications. The denominator derivatives are $2x \neq 0$ for $x \neq 0$ and then 2:

$$\lim_{x \rightarrow 0} \frac{\ln(1+x) - x}{x^2} = \lim_{x \rightarrow 0} \frac{1/(1+x) - 1}{2x} = \lim_{x \rightarrow 0} \frac{-1/(1+x)^2}{2} = -\frac{1}{2}.$$

(c) The form is infinity-over-infinity and the denominator derivative $2x$ is nonzero for $x > 0$:

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x^2} = \lim_{x \rightarrow \infty} \frac{1/x}{2x} = \lim_{x \rightarrow \infty} \frac{1}{2x^2} = 0.$$

6. Rolle's Theorem.

- (a) For $f(x) = x^3 - 3x$ on $[-2, 1]$, verify every hypothesis of Rolle's Theorem and find all $c \in (-2, 1)$ with $f'(c) = 0$.
- (b) Let $g(x) = |x - 1|$ on $[0, 2]$. Although $g(0) = g(2)$, there is no $c \in (0, 2)$ with $g'(c) = 0$. Explain why this does not contradict Rolle's Theorem.

Solution. (a) A polynomial is continuous on $[-2, 1]$ and differentiable on $(-2, 1)$; $f(-2) = f(1) = -2$. Since $f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1)$, the only required value is $c = -1$. The other zero, 1, is an endpoint and is excluded. (b) g is continuous with equal endpoint values but is not differentiable at 1. Its derivative is -1 for $x < 1$ and 1 for $x > 1$, never zero.

7. The Mean Value Theorem.

- (a) For $f(x) = \sqrt{x+1}$ on $[0, 8]$, verify the hypotheses of the Mean Value Theorem and find all $c \in (0, 8)$ satisfying its conclusion.
- (b) Suppose H is differentiable on $\mathbb{R}$ and $H'(x) > 0$ everywhere. Use the Mean Value Theorem to prove that H is strictly increasing: start with arbitrary $u < v$ and compare $H(v)$ with $H(u)$.

Solution. (a) f is continuous on $[0, 8]$ and differentiable on $(0, 8)$. The secant slope is $(3 - 1)/8 = 1/4$. Solving $1/(2\sqrt{c+1}) = 1/4$ gives $c = 3 \in (0, 8)$. (b) Differentiability implies continuity. The theorem on $[u, v]$ gives $H(v) - H(u) = H'(c)(v - u) > 0$ for some $c \in (u, v)$. Thus $H(u) < H(v)$ whenever $u < v$.

8. Derivative signs and horizontal tangents.

A differentiable function f on $\mathbb{R}$ satisfies

$$f'(x) = 15(x+1)x^2(x-1), \quad f(-1) = 2, \quad f(0) = 0, \quad f(1) = -2.$$

You do not need to find a formula for f .

- (a) Find all critical numbers and make a sign chart for f' . State intervals on which f is increasing or decreasing; you may list the intervals separated by critical numbers.
- (b) Classify each critical number as a local maximum, local minimum, or neither. Give each local extremum value and justify using derivative signs.
- (c) Find equations of all horizontal tangent lines. Which occurs without a local extremum?

Solution. (a) The critical numbers are $-1, 0, 1$. The signs on $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, \infty)$ are $+, -, -, +$, respectively. The function increases on the first and last intervals and decreases on the middle two. It also decreases across 0 on $(-1, 1)$. (b) The $+$ to $-$ change at -1 gives a local maximum of 2. The $-$ to $+$ change at 1 gives a local minimum of -2 . At 0, there is no sign change and no local extremum. (c) The lines are $y = 2$, $y = 0$, and $y = -2$. The horizontal tangent $y = 0$ at $x = 0$ occurs without a local extremum.

9. Derivative signs and a domain break.

Let $g(x) = \frac{x^2}{x-2}$, where $x \neq 2$.

- Find and factor $g'(x)$. Identify every critical number. Explain why $x = 2$ is not a critical number.
- Make a sign chart for g' and state the intervals of increase and decrease. Identify every local maximum and local minimum, giving both location and value.
- Find equations of all horizontal tangent lines. Explain why the two intervals of decrease separated by $x = 2$ cannot be combined into one interval of decrease.

Solution. (a) $g'(x) = [2x(x-2) - x^2]/(x-2)^2 = x(x-4)/(x-2)^2$. The critical numbers are 0 and 4. The number 2 is outside the domain. (b) The signs on $(-\infty, 0)$, $(0, 2)$, $(2, 4)$, and $(4, \infty)$ are $+, -, -, +$. The first and last intervals are intervals of increase, and the middle two are intervals of decrease. There is a local maximum of 0 at 0 and a local minimum of 8 at 4. (c) The lines are $y = 0$ and $y = 8$. An interval joining the two middle intervals would include 2, where g is undefined.

10. Local and absolute extrema.

Let $q(x) = x^4 - 2x^2$ on $[-2, 1]$. Reserve the term local extremum for interior points of this interval.

- Find all interior critical numbers. Factor $q'(x)$ and make a sign chart on the interval.
- Identify every interior local maximum and local minimum, giving both its location and its value. Find equations of the horizontal tangent lines at the interior critical numbers.
- Explain why the Extreme Value Theorem applies. Compare values at all interior critical numbers and both endpoints to find the absolute maximum and minimum. State every location where each is attained.

Solution. (a) $q'(x) = 4x(x-1)(x+1)$. The interior critical numbers are -1 and 0; 1 is an endpoint. The signs on $(-2, -1)$, $(-1, 0)$, and $(0, 1)$ are $-, +, -$. (b) There is a local minimum of -1 at -1 and a local maximum of 0 at 0. The corresponding

lines are $y = -1$ and $y = 0$. (c) The polynomial is continuous on the closed, bounded interval $[-2, 1]$. Compare

x	-2	-1	0	1
$q(x)$	8	-1	0	-1

The absolute maximum is 8 at -2 . The absolute minimum is -1 , attained at both -1 and 1 .