

# The Fundamental Theorem of Calculus

Math 140: Calculus with Analytic Geometry

April 13, 2026

## 1 Introduction

In the previous lecture, we evaluated the integral

$$\int_a^b e^x dx$$

by choosing sample points in a clever way so that the Riemann sum telescoped and produced

$$e^b - e^a.$$

This suggests a deeper connection between definite integrals and antiderivatives. In particular, if  $F'(x) = f(x)$ , then it appears that

$$\int_a^b f(x) dx = F(b) - F(a).$$

The goal of this lecture is to justify this relationship using the Mean Value Theorem.

## 2 Review of the Mean Value Theorem

We begin by recalling the Mean Value Theorem.

### Theorem (Mean Value Theorem)

Let  $f$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Then there exists a number  $c \in (a, b)$  such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

### Example

Let  $f(x) = x^2$  on  $[1, 3]$ . Then

$$\frac{f(3) - f(1)}{3 - 1} = \frac{9 - 1}{2} = 4.$$

Since  $f'(x) = 2x$ , we solve

$$2c = 4 \quad \Rightarrow \quad c = 2.$$

Thus, the Mean Value Theorem guarantees a point  $c = 2$  where the instantaneous rate of change equals the average rate of change.

### 3 Applying the Mean Value Theorem on Subintervals

Let  $F$  be a function such that  $F'(x) = f(x)$ . We divide  $[a, b]$  into subintervals of equal width  $\Delta x$  and define

$$x_i = a + i\Delta x.$$

On each subinterval  $[x_{i-1}, x_i]$ , the Mean Value Theorem guarantees a point  $c_i \in (x_{i-1}, x_i)$  such that

$$F'(c_i) = \frac{F(x_i) - F(x_{i-1})}{\Delta x}.$$

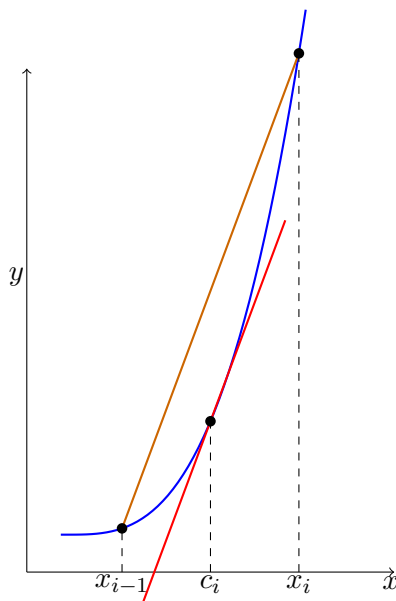
Since  $F'(c_i) = f(c_i)$ , this becomes

$$f(c_i) = \frac{F(x_i) - F(x_{i-1})}{\Delta x}.$$

Multiplying by  $\Delta x$ , we obtain

$$f(c_i)\Delta x = F(x_i) - F(x_{i-1}).$$

**Figure**



The point  $c_i$  is chosen so that the slope of the tangent line at  $c_i$  equals the slope of the secant line across the interval.

### 4 The Fundamental Theorem of Calculus

**Theorem (Fundamental Theorem of Calculus)**

Let  $f$  be continuous on  $[a, b]$ , and suppose  $F$  is an antiderivative of  $f$ , that is,  $F'(x) = f(x)$ . Then

$$\int_a^b f(x) dx = F(b) - F(a).$$

**Proof**

Divide  $[a, b]$  into  $n$  subintervals of equal width  $\Delta x$ . Then, as shown above, for each subinterval there exists  $c_i \in [x_{i-1}, x_i]$  such that

$$f(c_i)\Delta x = F(x_i) - F(x_{i-1}).$$

Summing over  $i = 1$  to  $n$ , we obtain

$$\sum_{i=1}^n f(c_i)\Delta x = \sum_{i=1}^n (F(x_i) - F(x_{i-1})).$$

The right-hand side is a telescoping sum:

$$\sum_{i=1}^n (F(x_i) - F(x_{i-1})) = F(x_n) - F(x_0) = F(b) - F(a).$$

Taking the limit as  $n \rightarrow \infty$ , the left-hand side becomes the definite integral:

$$\int_a^b f(x) dx = F(b) - F(a).$$

## 5 Examples

**Example**

Evaluate

$$\int_0^2 x^2 dx.$$

An antiderivative is  $F(x) = \frac{x^3}{3}$ . Thus,

$$\int_0^2 x^2 dx = \frac{2^3}{3} - \frac{0^3}{3} = \frac{8}{3}.$$

**Example**

Evaluate

$$\int_0^1 e^x dx.$$

An antiderivative is  $F(x) = e^x$ . Thus,

$$\int_0^1 e^x dx = e - 1.$$

## 6 Summary

The Fundamental Theorem of Calculus shows that definite integrals and antiderivatives are fundamentally connected. While Riemann sums define the integral as a limit of area approximations, the theorem shows that we can compute the exact value using antiderivatives:

$$\int_a^b f(x) dx = F(b) - F(a).$$

This result allows us to evaluate definite integrals efficiently without computing limits of sums.