Exponential and Logarithmic Functions

Math 140: Calculus with Analytic Geometry

Key Topics

- Exponential functions with base b
- Domain and range of exponential functions
- Logarithmic functions as inverses
- Properties of exponents and logarithms
- Solving exponential and logarithmic equations

1 Exponential Functions

Definition. *An* exponential function *is a function of the form*

$$f(x) = b^x,$$

where $b > 0$ *and* $b \neq 1$*.*

Two important examples are

$$f(x) = 2^x \qquad \text{and} \qquad g(x) = \left(\frac{1}{2}\right)^x.$$

Remark. *For any base* $b > 0$*, the domain of* b^x *is* $(-\infty, \infty)$ *and the range is* $(0, \infty)$*. In particular, exponential functions never take the value* 0*.*

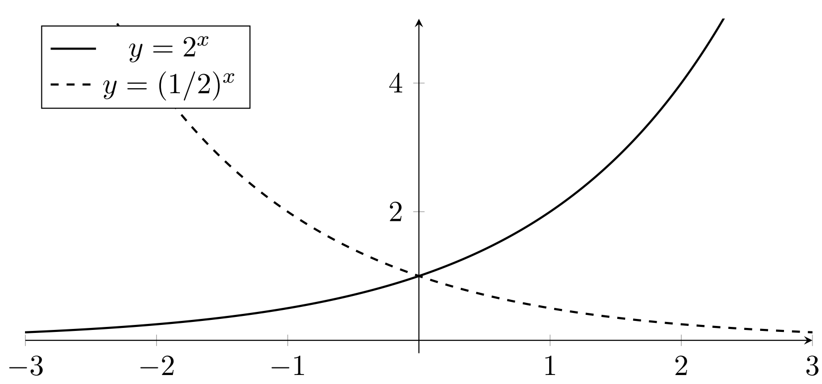

Figure 1: Exponential growth (2^x) and exponential decay $\left((1/2)^x\right)$.

Figure 1 shows the qualitative difference between growth and decay.

1.1 Properties of Exponents

For any real numbers x, y and any base $b > 0$, $b \neq 1$, the following properties hold:

$$b^{x+y} = b^x b^y, \qquad b^{x-y} = \frac{b^x}{b^y}, \qquad (b^x)^y = b^{xy}, \qquad b^0 = 1.$$

Example

Simplify

$$\frac{2^{x+3}}{2^{x-1}}.$$

Using exponent rules,

$$\frac{2^{x+3}}{2^{x-1}} = 2^{(x+3)-(x-1)} = 2^4 = 16.$$

2 Logarithmic Functions

Definition. *Let $b > 0$, $b \neq 1$. The* logarithmic function $\log_b(x)$ *is defined as the inverse of b^x. That is,*

$$\log_b(x) = y \quad \textit{if and only if} \quad b^y = x.$$

Remark. *Since b^x has domain $(-\infty, \infty)$ and range $(0, \infty)$, the logarithmic function $\log_b(x)$ has domain $(0, \infty)$ and range $(-\infty, \infty)$. Also, $\log_b(1) = 0$ since $b^0 = 1$.*

Examples

$$\log_2(8) = 3, \qquad \log_{1/2}\left(\tfrac{1}{4}\right) = 2.$$

3 Inverse Relationship Between Exponentials and Logarithms

3.1 Graphs and Inverses

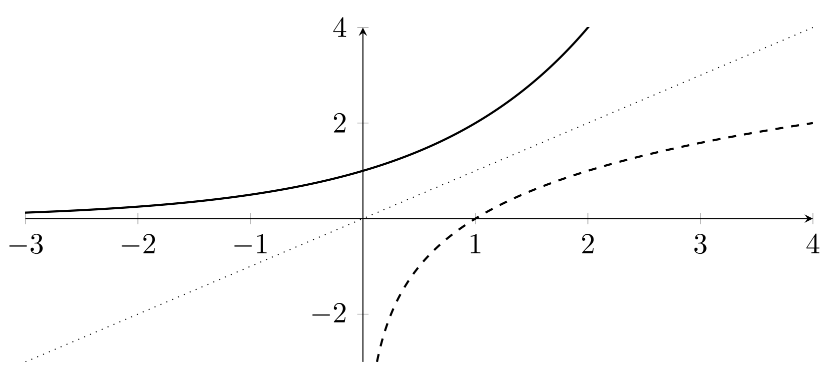

Figure 2: The functions $y = 2^x$ and $y = \log_2(x)$ are reflections across the line $y = x$.

Figure 2 illustrates that $y = \log_2(x)$ is the inverse of $y = 2^x$.
Figure 3 shows the same inverse relationship for a base between 0 and 1.

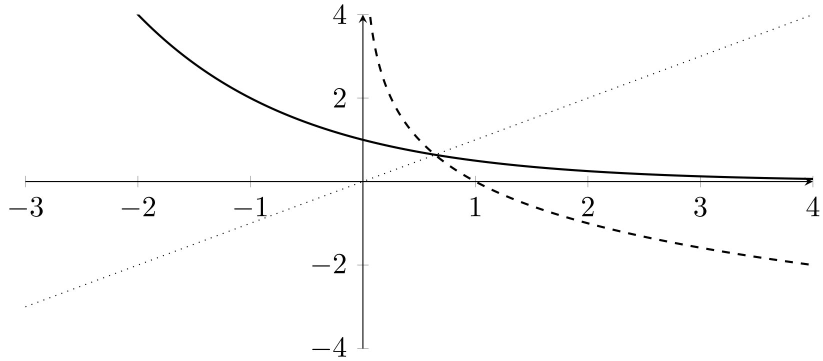

Figure 3: The functions $y = (1/2)^x$ and $y = \log_{1/2}(x)$ are reflections across $y = x$.

3.2 Properties of Logarithms

The properties of logarithms follow directly from the properties of exponents:

$$\log_b(xy) = \log_b(x) + \log_b(y), \qquad \log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y), \qquad \log_b(x^r) = r\log_b(x), \qquad \log_b(1) = 0.$$

Example

Simplify

$$\log_b\left(\frac{x^3}{\sqrt{y}}\right).$$

Using log properties,

$$\log_b(x^3) - \log_b(y^{1/2}) = 3\log_b(x) - \tfrac{1}{2}\log_b(y).$$

4 Solving Exponential and Logarithmic Equations

Example (Exponential)

Solve

$$2^{x+1} = 8.$$

Since $8 = 2^3$, we have $x + 1 = 3$, so $x = 2$.

Example (Logarithmic)

Solve

$$\log_2(x-1) + \log_2(x+1) = 3.$$

Combine logarithms:

$$\log_2\big((x-1)(x+1)\big) = 3 \Rightarrow x^2 - 1 = 8 \Rightarrow x = \pm 3.$$

Since $x - 1 > 0$ and $x + 1 > 0$, only $x = 3$ is valid.

5 Why This Matters for Calculus

- Exponential functions model growth and decay in many applications.
- Logarithms appear naturally when solving equations and, later, differential equations.
- The inverse relationship between exponentials and logarithms explains why their derivative rules are closely connected.