Differential Equations: Integrating Factors

Math 140: Calculus I

1 Introduction

In the previous lecture, we introduced differential equations and focused on separable equations. A separable differential equation can be rewritten so that all terms involving y appear on one side and all terms involving x appear on the other. This allowed us to solve the equation by integrating both sides.

In this lecture, we consider another important class of first-order differential equations. These equations are called first-order linear differential equations. They have the form

$$y' + p(x)y = q(x),$$

where $p(x)$ and $q(x)$ are functions of x.

Some first-order linear equations are separable, but many are not. The goal of this lecture is to develop a method that works for every first-order linear differential equation. This method is called the method of integrating factors.

2 First-Order Linear Differential Equations

A first-order linear differential equation has the form

$$y' + p(x)y = q(x).$$

The word linear means that the unknown function y and its derivative y' appear only to the first power and are not multiplied together.

For example, the equation

$$y' + 2y = e^x$$

is linear, while

$$y' = y^2$$

is not linear.

The equation

$$y' + \frac{1}{x}y = x^2$$

is also linear, provided we work on a domain where $x \neq 0$. For example, we might work on the interval $x > 0$.

3 Motivation for the Integrating Factor

The difficulty with a linear equation

$$y' + p(x)y = q(x)$$

is that the left side is close to a product rule, but it is not obviously the derivative of a product.

Recall that

$$\frac{d}{dx}\big(\mu(x)y\big) = \mu(x)y' + \mu'(x)y.$$

If we could multiply the differential equation by a function $\mu(x)$ so that the left side becomes this product derivative, then the equation would become much easier to solve.

Multiplying

$$y' + p(x)y = q(x)$$

by $\mu(x)$ gives

$$\mu(x)y' + \mu(x)p(x)y = \mu(x)q(x).$$

We want the left side to match

$$\mu(x)y' + \mu'(x)y.$$

This will happen if

$$\mu'(x) = \mu(x)p(x).$$

This equation determines the integrating factor.

4 Deriving the Integrating Factor

We now solve

$$\mu'(x) = p(x)\mu(x).$$

This equation is separable:

$$\frac{\mu'(x)}{\mu(x)} = p(x).$$

Equivalently,

$$\frac{1}{\mu}\,d\mu = p(x)\,dx.$$

Integrating both sides gives

$$\ln|\mu| = \int p(x)\,dx.$$

Exponentiating both sides gives

$$\mu(x) = e^{\int p(x)\,dx}.$$

Thus, for the linear equation

$$y' + p(x)y = q(x),$$

an integrating factor is

$$\mu(x) = e^{\int p(x)\,dx}.$$

5 Why the Integrating Factor Works

We now show why this method works in general.

Start with the linear equation

$$y' + p(x)y = q(x).$$

Let

$$\mu(x) = e^{\int p(x)\,dx}.$$

Then

$$\mu'(x) = p(x)\mu(x).$$

Multiplying the differential equation by $\mu(x)$ gives

$$\mu(x)y' + \mu(x)p(x)y = \mu(x)q(x).$$

Since $\mu'(x) = p(x)\mu(x)$, the left side becomes

$$\mu(x)y' + \mu'(x)y.$$

By the product rule,

$$\mu(x)y' + \mu'(x)y = \frac{d}{dx}\big(\mu(x)y\big).$$

Therefore, the differential equation becomes

$$\frac{d}{dx}\big(\mu(x)y\big) = \mu(x)q(x).$$

Integrating both sides gives

$$\mu(x)y = \int \mu(x)q(x)\,dx.$$

Finally, solving for y gives the solution.

The important idea is that the integrating factor turns the left side of the differential equation into the derivative of a product.

6 Summary of the Method

To solve a first-order linear differential equation

$$y' + p(x)y = q(x),$$

we use the following steps.

1. Identify $p(x)$ and $q(x)$.
2. Compute the integrating factor
$$\mu(x) = e^{\int p(x)\,dx}.$$
3. Multiply the entire equation by $\mu(x)$.
4. Rewrite the left side as
$$\frac{d}{dx}\big(\mu(x)y\big).$$
5. Integrate both sides and solve for y.

7 Examples

Example

Solve the differential equation

$$y' + 2y = e^x.$$

Here,

$$p(x) = 2, \qquad q(x) = e^x.$$

The integrating factor is

$$\mu(x) = e^{\int 2\,dx} = e^{2x}.$$

Multiplying the differential equation by e^{2x} gives

$$e^{2x}y' + 2e^{2x}y = e^{3x}.$$

The left side is the derivative of $e^{2x}y$, so

$$\frac{d}{dx}\left(e^{2x}y\right) = e^{3x}.$$

Integrating both sides gives

$$e^{2x}y = \frac{1}{3}e^{3x} + C.$$

Solving for y gives

$$y = \frac{1}{3}e^x + Ce^{-2x}.$$

Example

Solve the initial value problem

$$y' + 3y = 6, \qquad y(0) = 1.$$

Here,

$$p(x) = 3, \qquad q(x) = 6.$$

The integrating factor is

$$\mu(x) = e^{\int 3\,dx} = e^{3x}.$$

Multiplying the equation by e^{3x} gives

$$e^{3x}y' + 3e^{3x}y = 6e^{3x}.$$

Thus,

$$\frac{d}{dx}\left(e^{3x}y\right) = 6e^{3x}.$$

Integrating both sides gives

$$e^{3x}y = 2e^{3x} + C.$$

Solving for y gives

$$y = 2 + Ce^{-3x}.$$

Applying the initial condition $y(0) = 1$ gives $C = -1$, so

$$y = 2 - e^{-3x}.$$

Example

Solve the differential equation

$$y' + \frac{1}{x}y = x^2, \qquad x > 0.$$

Here,

$$p(x) = \frac{1}{x}, \qquad q(x) = x^2.$$

The integrating factor is

$$\mu(x) = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x.$$

Multiplying the equation by x gives

$$xy' + y = x^3.$$

The left side is the derivative of xy, so

$$\frac{d}{dx}(xy) = x^3.$$

Integrating both sides gives

$$xy = \frac{x^4}{4} + C.$$

Solving for y gives

$$y = \frac{x^3}{4} + \frac{C}{x}.$$

Example

Solve the initial value problem

$$y' - \frac{2}{x}y = x^2, \qquad y(1) = 4, \qquad x > 0.$$

Here,

$$p(x) = -\frac{2}{x}, \qquad q(x) = x^2.$$

The integrating factor is

$$\mu(x) = e^{\int -\frac{2}{x}\,dx} = e^{-2\ln x} = x^{-2}.$$

Multiplying the equation by x^{-2} gives

$$x^{-2}y' - 2x^{-3}y = 1.$$

The left side is the derivative of $x^{-2}y$, so

$$\frac{d}{dx}\left(x^{-2}y\right) = 1.$$

Integrating both sides gives

$$x^{-2}y = x + C.$$

Solving for y gives

$$y = x^3 + Cx^2.$$

Applying the initial condition $y(1) = 4$ gives $C = 3$, so

$$y = x^3 + 3x^2.$$

Example

Solve the initial value problem

$$y' + y = e^{-x}, \qquad y(0) = 2.$$

Here,

$$p(x) = 1, \qquad q(x) = e^{-x}.$$

The integrating factor is

$$\mu(x) = e^{\int 1\,dx} = e^{x}.$$

Multiplying the equation by e^x gives

$$e^x y' + e^x y = 1.$$

Thus,

$$\frac{d}{dx}\left(e^x y\right) = 1.$$

Integrating both sides gives

$$e^x y = x + C.$$

Solving for y gives

$$y = (x + C)e^{-x}.$$

Applying the initial condition $y(0) = 2$ gives $C = 2$, so

$$y = (x + 2)e^{-x}.$$

Example

Solve the initial value problem

$$y' + 2xy = 2x, \qquad y(0) = 3.$$

Here,

$$p(x) = 2x, \qquad q(x) = 2x.$$

The integrating factor is

$$\mu(x) = e^{\int 2x\,dx} = e^{x^2}.$$

Multiplying the equation by e^{x^2} gives

$$e^{x^2} y' + 2xe^{x^2} y = 2xe^{x^2}.$$

The left side is the derivative of $e^{x^2}y$, so

$$\frac{d}{dx}\left(e^{x^2} y\right) = 2xe^{x^2}.$$

Integrating both sides gives

$$e^{x^2} y = e^{x^2} + C.$$

Solving for y gives

$$y = 1 + Ce^{-x^2}.$$

Applying the initial condition $y(0) = 3$ gives $C = 2$, so

$$y = 1 + 2e^{-x^2}.$$

8 Summary

First-order linear differential equations have the form

$$y' + p(x)y = q(x).$$

When such an equation is not separable, we can still solve it using an integrating factor.

The integrating factor is

$$\mu(x) = e^{\int p(x)\,dx}.$$

Multiplying the differential equation by this function turns the left side into the derivative of a product:

$$\frac{d}{dx}\big(\mu(x)y\big) = \mu(x)q(x).$$

This allows us to integrate both sides and solve for y.

The key idea is that the integrating factor is chosen specifically so that the product rule applies.