Differential Equations

Math 140: Calculus I

1 Introduction

Up to this point in the course, we have studied derivatives and antiderivatives. Derivatives allow us to compute rates of change, while antiderivatives allow us to recover a function from its derivative. In particular, given a function $f(x)$, we have learned how to find a function $F(x)$ such that

$$F'(x) = f(x).$$

This leads to the concept of the indefinite integral and families of antiderivatives.

In this lecture, we consider a broader question. Instead of being given a function and asked to differentiate or integrate it, we are given a relationship involving an unknown function and its derivative, and we are asked to determine the function itself. This leads to the study of differential equations.

Example

Consider the equation

$$\frac{dy}{dx} = 2x.$$

We are asked to find a function $y(x)$ whose derivative is $2x$. From our knowledge of antiderivatives, we know that

$$y = x^2 + C.$$

This is not a single function, but rather a family of functions, each differing by a constant.

Now suppose we are given an additional condition, such as

$$y(1) = 3.$$

Substituting into the general solution gives

$$3 = 1^2 + C,$$

so $C = 2$. Therefore, the unique function satisfying both the differential equation and the initial condition is

$$y = x^2 + 2.$$

This example illustrates two important ideas. First, differential equations typically have infinitely many solutions. Second, an additional condition, called an initial condition, allows us to determine a unique solution.

2 Differential Equations

A differential equation is an equation that relates a function to one or more of its derivatives.

The following are examples of differential equations:

$$\frac{dy}{dx} = x^2, \qquad \frac{d^2y}{dx^2} + y = 0, \qquad y' + y = e^x.$$

The order of a differential equation is the highest derivative that appears in the equation.

2.1 Linear and Nonlinear Equations

We now restrict our attention to first-order differential equations. A first-order differential equation is called linear if it can be written in the form

$$y' + p(x)y = q(x),$$

where $p(x)$ and $q(x)$ are functions of x.

The equation

$$y' + 3y = x$$

is linear, while

$$y' = y^2$$

is nonlinear, since it involves a nonlinear power of y.

Example

Consider the differential equation

$$y' + \frac{1}{x}y = x^2, \qquad x > 0.$$

We verify that the function

$$y = \frac{x^3}{4}$$

satisfies this equation.

Computing the derivative gives

$$y' = \frac{3x^2}{4}.$$

Substituting into the differential equation yields

$$y' + \frac{1}{x}y = \frac{3x^2}{4} + \frac{1}{x} \cdot \frac{x^3}{4} = \frac{3x^2}{4} + \frac{x^2}{4} = x^2.$$

Since both sides agree, the function satisfies the differential equation on the given domain.

3 Separable Differential Equations

A first-order differential equation is called separable if it can be written in the form

$$\frac{dy}{dx} = f(x)g(y),$$

so that variables can be separated and integrated.

Example

Solve the initial value problem

$$\frac{dy}{dx} = xy, \qquad y(0) = 2, \qquad -\infty < x < \infty.$$

Separating variables gives

$$\frac{1}{y}dy = x\,dx.$$

Integrating both sides produces

$$\ln|y| = \frac{x^2}{2} + C.$$

Solving for y yields

$$y = Ce^{x^2/2}.$$

Applying the initial condition $y(0) = 2$ gives $C = 2$, so the solution is

$$y = 2e^{x^2/2}.$$

This equation is both separable and linear.

Example

Solve

$$\frac{dy}{dx} = \frac{x}{1+y}, \qquad y(0) = 0, \qquad -\infty < x < \infty.$$

Separating variables gives

$$(1+y)dy = xdx.$$

Integrating both sides produces

$$y + \frac{y^2}{2} = \frac{x^2}{2} + C.$$

Applying the initial condition gives $C = 0$, so the solution is

$$y + \frac{y^2}{2} = \frac{x^2}{2}.$$

This equation is separable but nonlinear.

Example

Solve the initial value problem

$$\frac{dy}{dx} = \frac{1+y}{x}, \qquad y(1) = 1, \qquad x > 0.$$

Separating variables gives

$$\frac{1}{1+y}\,dy = \frac{1}{x}\,dx.$$

Integrating both sides produces

$$\ln|1+y| = \ln|x| + C.$$

Exponentiating both sides yields

$$|1+y| = Cx.$$

Solving for y gives

$$y = Cx - 1.$$

Applying the initial condition $y(1) = 1$ gives

$$1 = C(1) - 1,$$

so $C = 2$. Therefore, the solution is

$$y = 2x - 1.$$

This equation is both separable and linear.

Example

Solve

$$\frac{dy}{dx} = x(1+y^2), \qquad y(0) = 0, \qquad -\infty < x < \infty.$$

Separating variables gives

$$\frac{1}{1+y^2} dy = x dx.$$

Integrating both sides produces

$$\arctan(y) = \frac{x^2}{2} + C.$$

Applying the initial condition gives $C = 0$, so

$$\arctan(y) = \frac{x^2}{2}.$$

Solving for y yields

$$y = \tan\left(\frac{x^2}{2}\right).$$

This equation is separable but nonlinear. The solution is defined only where the tangent function is defined, so the domain must exclude points where $\frac{x^2}{2} = \frac{\pi}{2} + k\pi$.

4 Summary

Solving separable differential equations reduces to evaluating integrals. Some separable equations are also linear, while others are nonlinear. In all cases, the general solution contains a constant, and initial conditions determine a unique solution. Domains must also be considered, as solutions may not be valid for all values of x.