Curve Sketching

Math 140: Calculus with Analytic Geometry

1 Introduction

Throughout this course we have developed several tools for understanding functions. These tools allow us to determine the local and global behavior of functions without relying solely on technology.

Important ideas include:

- Limits and continuity
- Infinite limits and vertical asymptotes
- Limits at infinity and horizontal asymptotes
- The First Derivative Test
- The Second Derivative Test

We now combine these ideas to sketch graphs of functions using calculus.

Curve sketching allows us to determine the important features of a graph, including its extrema, asymptotic behavior, and concavity.

2 Important Features of Graphs

When analyzing a function we typically look for the following features.

- Domain
- Intercepts
- Vertical asymptotes
- Horizontal asymptotes

- Critical numbers
- Increasing and decreasing behavior
- Local extrema
- Concavity
- Inflection points

Together these features determine the overall shape of the graph.

3 Procedure for Curve Sketching

To sketch the graph of a function $f(x)$, analyze the following steps.

1. Determine the **domain**.
2. Find the **intercepts**.
3. Identify any **vertical asymptotes** using infinite limits.
4. Determine **horizontal asymptotes** using limits at infinity.
5. Compute the first derivative $f'(x)$.
6. Find **critical numbers** where

$$f'(x) = 0 \quad \text{or} \quad f'(x) \text{ is undefined.}$$

7. Use the **First Derivative Test** to determine intervals of increase and decrease.
8. Compute the second derivative $f''(x)$.
9. Use the **Second Derivative Test** to determine concavity.
10. Identify any **inflection points**.
11. Combine this information to sketch the graph.

4 Example 1: Polynomial Function

Sketch the graph of

$$f(x) = x^3 - 3x.$$

Step 1: Domain

Polynomials are defined for all real numbers.

$$(-\infty, \infty)$$

Step 2: First Derivative

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1)$$

Critical numbers:

$$x = \pm 1$$

Step 3: Increasing/Decreasing

- Increasing on $(-\infty, -1)$
- Decreasing on $(-1, 1)$
- Increasing on $(1, \infty)$

Thus

$$\text{local maximum at } x = -1$$

$$\text{local minimum at } x = 1$$

Step 4: Second Derivative

$$f''(x) = 6x$$

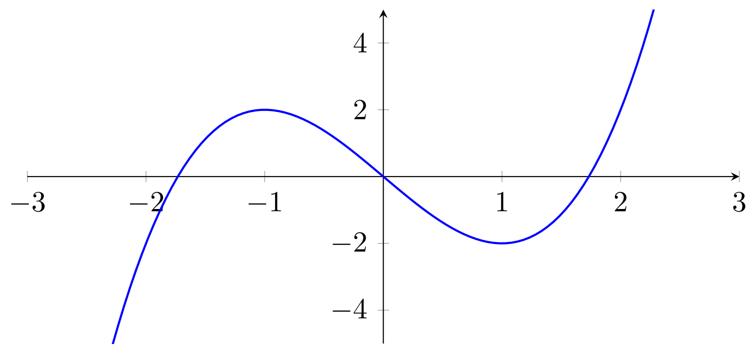
Inflection point:

$$x = 0$$

5 Example 2: Rational Function

Sketch the graph of

$$f(x) = \frac{1}{x-1}.$$



Domain

$$x \neq 1$$

Vertical Asymptote

$$x = 1$$

Horizontal Asymptote

$$\lim_{x \to \pm\infty} \frac{1}{x-1} = 0$$

Thus

$$y = 0$$

Derivative

$$f'(x) = -\frac{1}{(x-1)^2} < 0$$

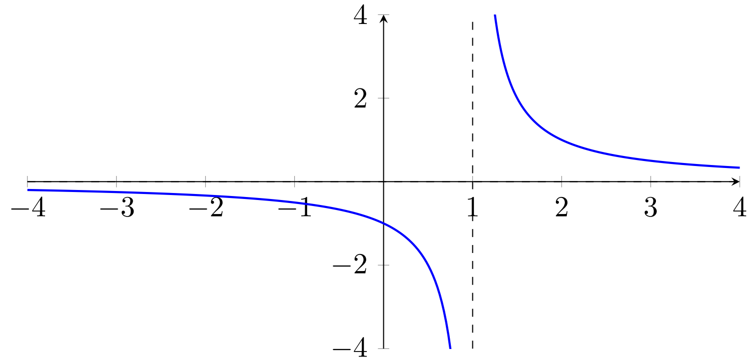
Thus the function is decreasing on both intervals.

6 Example 3: Trigonometric Function

Sketch the graph of

$$f(x) = x \sin x$$

on the interval $[-2\pi, 2\pi]$.



First Derivative

Using the product rule,

$$f'(x) = \sin x + x \cos x$$

Critical numbers occur when

$$\sin x + x \cos x = 0$$

or

$$\tan x = -x$$

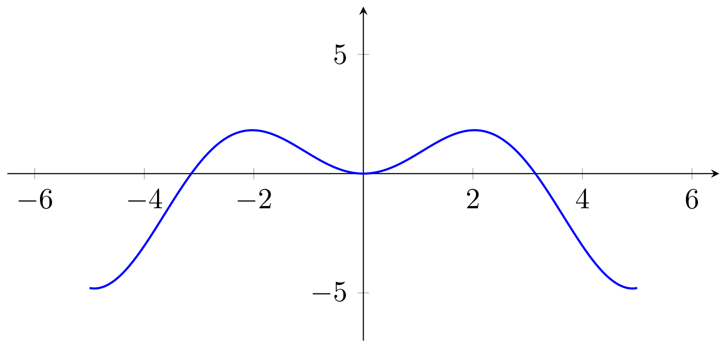
which must be solved numerically.

Second Derivative

$$f''(x) = 2 \cos x - x \sin x$$

Observations

- Zeros occur at $x = n\pi$
- The oscillations increase in magnitude as $|x|$ increases



7 Example 4: Logarithmic and Exponential Behavior

Sketch the graph of

$$f(x) = e^{-x} \ln x$$

for $x > 0$.

Domain

$$(0, \infty)$$

Vertical Asymptote

$$\lim_{x \to 0^+} e^{-x} \ln x = -\infty$$

Thus

$$x = 0$$

is a vertical asymptote.

Horizontal Asymptote

$$\lim_{x \to \infty} e^{-x} \ln x = \lim_{x \to \infty} \frac{\ln x}{e^x}$$

Using L'Hôpital's Rule,

$$= 0$$

Thus

$$y = 0$$

is a horizontal asymptote.

First Derivative

$$f'(x) = e^{-x}\left(\frac{1}{x} - \ln x\right)$$

Critical number occurs when

$$\frac{1}{x} = \ln x$$

which occurs near

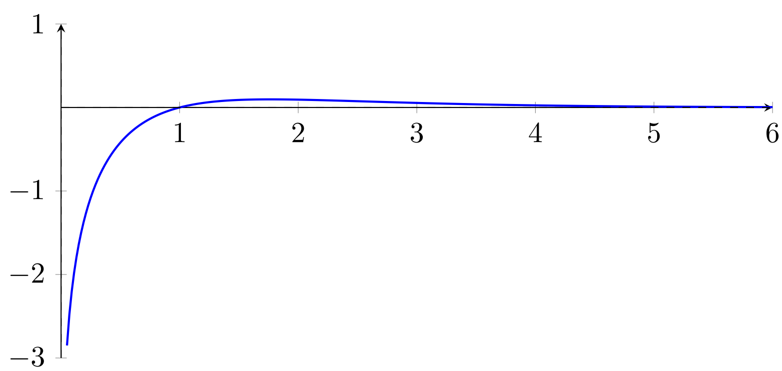
$$x \approx 1.76$$

Thus the function has a local maximum near this point.

Second Derivative

$$f''(x) = e^{-x}\left(\ln x - \frac{2}{x} - \frac{1}{x^2}\right)$$

which determines concavity.



8 Summary

Curve sketching combines several fundamental ideas from calculus.

- Limits describe behavior near asymptotes.
- The first derivative determines increasing and decreasing behavior.
- The second derivative determines concavity.
- Critical numbers reveal potential extrema.
- Asymptotes describe the behavior of functions near the boundaries of their domain.

Using these tools together allows us to understand the global shape of functions and to sketch accurate graphs.