Continuity

Math 140: Calculus with Analytic Geometry

Key Topics

- Definition of continuity at a point
- Continuity of elementary and transcendental functions
- Verifying continuity using limits
- Continuity of piecewise-defined functions
- Continuity of compositions
- Continuity of inverse functions

1 Continuity at a Point

We now formalize the intuitive idea that a function has no "breaks" at a point.

Definition. *A function $f(x)$ is said to be* continuous at $x = c$ *if all three of the following conditions hold:*

1. *$f(c)$ is defined,*
2. *$\lim_{x \to c} f(x)$ exists,*
3. *$\lim_{x \to c} f(x) = f(c)$.*

If a function is continuous at every point in an interval, we say it is continuous on that interval.

2 Continuity of Common Functions

From the limit results established earlier, we record the following facts.

Theorem 1. *Each of the following functions is continuous at every point in its domain:*

- *polynomials,*
- *rational functions,*
- $\sin(x)$, $\cos(x)$, *and* $\tan(x)$,
- *exponential functions* b^x,
- *logarithmic functions* $\log_b(x)$.

Remark. *The continuity of trigonometric, exponential, and logarithmic functions was already implicitly used when evaluating limits of transcendental functions.*

3 Verifying Continuity Using the Definition

We now practice verifying continuity by checking the three defining conditions.

Example 1: A Rational Function

Consider

$$f(x) = \frac{x^2+1}{x-1}.$$

The function is defined for all $x \neq 1$. Let $c \neq 1$.

- $f(c)$ is defined,
- $\lim_{x \to c} \frac{x^2+1}{x-1} = \frac{c^2+1}{c-1}$,
- the limit equals $f(c)$.

Thus, f is continuous at every $c \neq 1$.

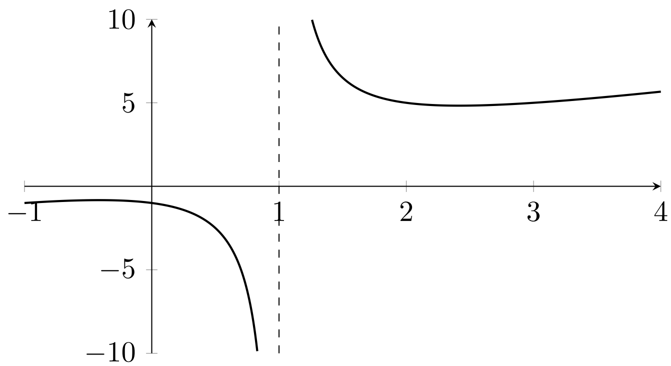

Figure 1: The rational function $f(x) = \frac{x^2+1}{x-1}$ is continuous at every point in its domain.

4 Continuity of Piecewise Functions

For piecewise-defined functions, continuity must be checked carefully at the transition points.

Example 2: The Absolute Value Function

Define

$$f(x) = |x| = \begin{cases} -x, & x < 0, \\ x, & x \geq 0. \end{cases}$$

At $x = 0$:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-x) = 0, \qquad \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0,$$

and $f(0) = 0$. Therefore, f is continuous at $x = 0$.

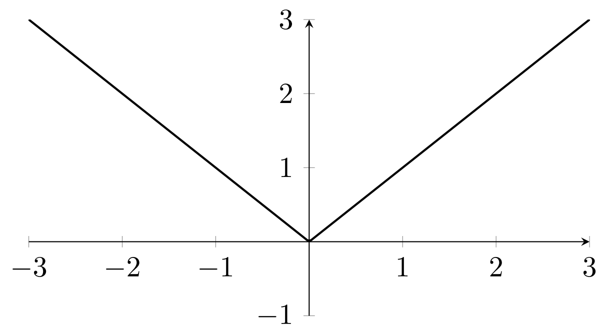

Figure 2: The graph of $f(x) = |x|$, which is continuous everywhere.

Example 3: A Continuous Extension of $\sin(x)/x$

Define

$$f(x) = \begin{cases} \dfrac{\sin(x)}{x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

Since

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1 = f(0),$$

the function f is continuous at $x = 0$.

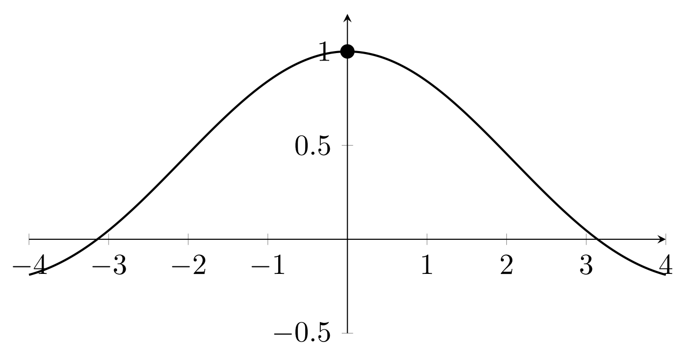

Figure 3: A continuous extension of $\sin(x)/x$ obtained by defining $f(0) = 1$.

Example 4: A Discontinuous Variant

Define

$$g(x) = \begin{cases} \dfrac{\sin(x)}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Although the limit $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$ exists, we have $g(0) = 0 \neq 1$. Thus, g is *not* continuous at $x = 0$.

5 Continuity of Compositions

Theorem 2 (Continuity of Compositions)**.** *If f is continuous at c and g is continuous at $f(c)$, then the composition*

$$(g \circ f)(x) = g(f(x))$$

is continuous at c.

Example

Let $f(x) = x^2$ and $g(x) = \sin(x)$. Since both functions are continuous everywhere, the composition

$$(g \circ f)(x) = \sin(x^2)$$

is continuous for all x.

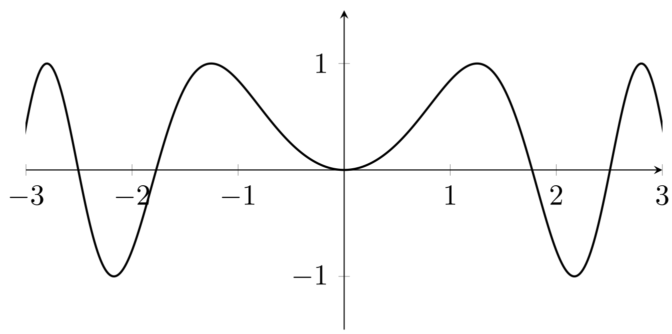

Figure 4: The function $y = \sin(x^2)$ is continuous as a composition of continuous functions.

6 Continuity of Inverse Functions

Theorem 3 (Continuity of Inverse Functions)**.** *If f is continuous and one-to-one on an interval, then its inverse function f^{-1} is continuous on its domain.*

Example

Consider $f(x) = x^3$. This function is continuous and one-to-one on $\mathbb{R}$, so its inverse $f^{-1}(x) = \sqrt[3]{x}$ is continuous on $\mathbb{R}$.

7 Why This Matters for Calculus

- Continuity guarantees that limits behave predictably.
- Many theorems in calculus require continuity as a hypothesis.
- Understanding continuity prepares us for the Intermediate Value Theorem and derivative theory.

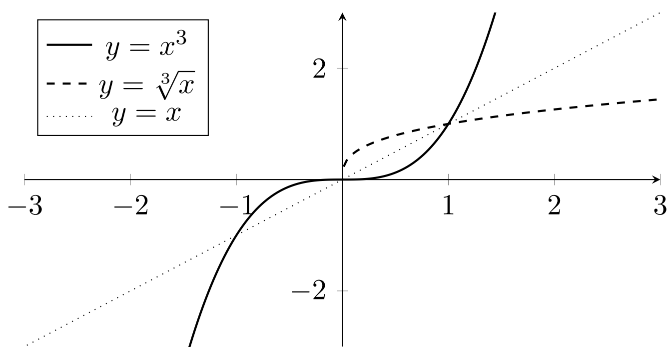

Figure 5: The function $y = x^3$ and its inverse $y = \sqrt[3]{x}$ are both continuous.