

# Basic Derivative Rules

Math 140: Calculus with Analytic Geometry

## Key Topics

- Linearity Rules
- Product and Power Rules
- Interpreting derivatives as slopes and rates of change

## 1 Why We Need Derivative Rules

The limit definition

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

allows us to compute derivatives from first principles, but repeating this limit calculation for every new function quickly becomes impractical. In this lecture we develop a small collection of derivative rules that let us compute derivatives efficiently.

## 2 Linearity Rules

The next two rules explain how derivatives behave with respect to addition and constant multiples.

**Theorem 2.1** (Constant Multiple Rule). *If  $f(x) = c \cdot g(x)$  where  $c$  is a constant and  $g$  is differentiable, then*

$$f'(x) = c \cdot g'(x).$$

*Proof.* Fix  $a$ . Using the limit definition and factoring out the constant  $c$ ,

$$\begin{aligned} f'(a) &= \lim_{h \rightarrow 0} \frac{c \cdot g(a+h) - c \cdot g(a)}{h} = \lim_{h \rightarrow 0} c \frac{g(a+h) - g(a)}{h} \\ &= c \lim_{h \rightarrow 0} \frac{g(a+h) - g(a)}{h} = c \cdot g'(a). \end{aligned}$$

□

**Theorem 2.2** (Sum and Difference Rules). *If  $f(x) = g(x) \pm h(x)$  and both  $g$  and  $h$  are differentiable, then*

$$f'(x) = g'(x) \pm h'(x).$$

*Proof.* Fix  $a$ . Using the limit definition and limit laws,

$$\begin{aligned} f'(a) &= \lim_{h \rightarrow 0} \frac{(g(a+h) \pm h(a+h)) - (g(a) \pm h(a))}{h} \\ &= \lim_{h \rightarrow 0} \left( \frac{g(a+h) - g(a)}{h} \pm \frac{h(a+h) - h(a)}{h} \right) \\ &= \lim_{h \rightarrow 0} \frac{g(a+h) - g(a)}{h} \pm \lim_{h \rightarrow 0} \frac{h(a+h) - h(a)}{h} \\ &= g'(a) \pm h'(a). \end{aligned}$$

□

**Example 2.1.** Differentiate  $f(x) = 3x^2 + 4x + 5$ .

Applying the constant multiple and addition rules, we find that

$$f'(x) = 3(2x) + 4(1) + 0 = 6x + 4.$$

### 3 Product and Power Rules

**Theorem 3.1** (Product Rule). If  $f(x) = u(x)v(x)$ , where both  $u$  and  $v$  are differentiable, then

$$f'(x) = u'(x)v(x) + u(x)v'(x).$$

*Proof.* Fix  $a$ . Using the limit definition,

$$f'(a) = \lim_{h \rightarrow 0} \frac{u(a+h)v(a+h) - u(a)v(a)}{h}.$$

Add and subtract  $u(a+h)v(a)$  in the numerator:

$$\frac{u(a+h)v(a+h) - u(a+h)v(a) + u(a+h)v(a) - u(a)v(a)}{h}.$$

Group terms:

$$= \frac{u(a+h)(v(a+h) - v(a))}{h} + \frac{v(a)(u(a+h) - u(a))}{h}.$$

Taking limits gives

$$f'(a) = u(a)v'(a) + v(a)u'(a).$$

□

**Example 3.1.** Differentiate  $f(x) = x^2(x^3 - 1)$ .

Let  $u(x) = x^2$  and  $v(x) = x^3 - 1$ . Then

$$f'(x) = 2x(x^3 - 1) + x^2(3x^2) = 5x^4 - 2x.$$

**Theorem 3.2** (Power Rule). If  $f(x) = x^n$  where  $n$  is a positive integer, then

$$f'(x) = nx^{n-1}.$$

*Proof.* First, we show that the power rule holds when  $n = 1$ . In this case,  $f(x) = x$  and we know that  $f'(x) = 1 = 1x^0$ . Next, let  $n \geq 1$  and suppose that the power rule holds for  $x^n$ . Define  $f(x) = x^{n+1} = x \cdot x^n$ . Then, the product rule implies that

$$f'(x) = 1x^n + x(nx^{n-1}) = (n+1)x^n.$$

Therefore, the power rule holds for all positive integers.

□

**Example 3.2.** Differentiate  $f(x) = x^5$ .

By the power rule,

$$f'(x) = 5x^4.$$

**Example 3.3.** Differentiate  $f(x) = \sqrt[3]{x} = x^{1/3}$ .

We will later justify the power rule for negative and rational exponents using the quotient rule and implicit differentiation. A complete justification for arbitrary real exponents will be given after logarithmic differentiation is introduced. For now, we record the expected result:

$$f'(x) = \frac{1}{3}x^{-2/3}.$$

## 4 Differentiating Polynomials

Combining the constant rule, power rule, constant multiple rule, and sum/difference rules allows us to differentiate polynomials quickly.

**Example 4.1.** Differentiate  $p(x) = 4x^6 - 3x^2 + 7x - 9$ .

$$p'(x) = 24x^5 - 6x + 7.$$

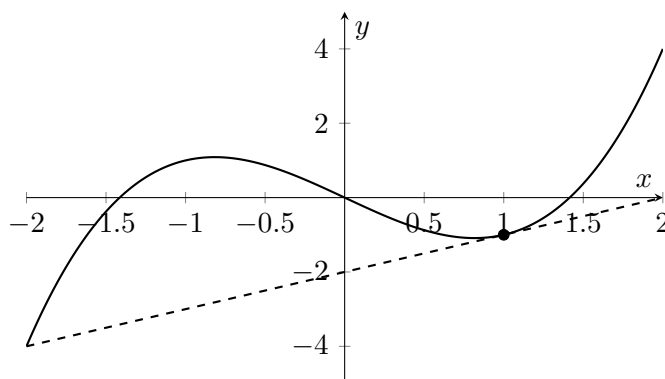
**Example 4.2.** Find the equation of the tangent line to  $y = x^3 - 2x$  at  $x = 1$ .

First compute the derivative:

$$y' = 3x^2 - 2.$$

At  $x = 1$ , the slope is  $m = 3(1)^2 - 2 = 1$ . The point on the curve is  $(1, 1^3 - 2 \cdot 1) = (1, -1)$ . Thus the tangent line is

$$y + 1 = 1(x - 1) \implies y = x - 2.$$



## 5 Why This Matters for Calculus

- Derivative rules let us compute slopes and rates of change quickly without redoing limit calculations.
- The power rule and linearity rules are the foundation for differentiating polynomials and many other functions.
- Efficient differentiation is essential for optimization, graphing, and motion problems.