Math 140: Calculus I (Spring 2026)
Homework 14 Solutions

1. Evaluate the indefinite integral:

$$\int (x+2)\sqrt{x-1}\,dx.$$

Let

$$u = x - 1.$$

Then

$$x = u + 1, \qquad x + 2 = u + 3, \qquad dx = du.$$

Thus,

$$\int (x+2)\sqrt{x-1}\,dx = \int (u+3)u^{1/2}\,du = \int \left(u^{3/2} + 3u^{1/2}\right)\,du.$$

Therefore,

$$= \frac{2}{5}u^{5/2} + 2u^{3/2} + C.$$

Substituting back,

$$\int (x+2)\sqrt{x-1}\,dx = \frac{2}{5}(x-1)^{5/2} + 2(x-1)^{3/2} + C.$$

2. Evaluate the definite integral:

$$\int_0^{\pi/3} \sin^3(x)\,dx.$$

Write

$$\sin^3 x = \sin x(1 - \cos^2 x).$$

Let

$$u = \cos x, \qquad du = -\sin x\,dx.$$

When $x = 0$, we have $u = 1$, and when $x = \pi/3$, we have $u = 1/2$. Thus,

$$\int_0^{\pi/3} \sin^3(x)\,dx = -\int_1^{1/2} (1 - u^2)\,du = \int_{1/2}^1 (1 - u^2)\,du.$$

Now compute:

$$= \left(u - \frac{u^3}{3}\right)\Big|_{1/2}^{1} = \left(1 - \frac{1}{3}\right) - \left(\frac{1}{2} - \frac{1}{24}\right) = \frac{2}{3} - \frac{11}{24} = \frac{5}{24}.$$

3. Evaluate the indefinite integral:

$$\int \frac{x}{1+x^4}\,dx.$$

Let

$$u = x^2, \qquad du = 2x\,dx.$$

Then

$$x\,dx = \frac{1}{2}\,du, \qquad 1+x^4 = 1+u^2.$$

Thus,

$$\int \frac{x}{1+x^4}\,dx = \frac{1}{2}\int \frac{1}{1+u^2}\,du = \frac{1}{2}\arctan(u) + C.$$

Substituting back,

$$\int \frac{x}{1+x^4}\,dx = \frac{1}{2}\arctan(x^2) + C.$$

4. Evaluate the definite integral:

$$\int_1^3 \frac{2x}{x^2+1}\,dx.$$

Let

$$u = x^2+1, \qquad du = 2x\,dx.$$

When $x = 1$, we have $u = 2$, and when $x = 3$, we have $u = 10$. Thus,

$$\int_1^3 \frac{2x}{x^2+1}\,dx = \int_2^{10} \frac{1}{u}\,du = \ln|u|\Big|_2^{10} = \ln 10 - \ln 2 = \ln 5.$$

5. Evaluate the indefinite integral:

$$\int \frac{5x+4}{x^2+2x-3}\,dx.$$

Factor the denominator:

$$x^2+2x-3 = (x+3)(x-1).$$

Use partial fractions:

$$\frac{5x+4}{(x+3)(x-1)} = \frac{A}{x+3} + \frac{B}{x-1}.$$

Then

$$5x+4 = A(x-1) + B(x+3).$$

Setting $x = 1$ gives

$$9 = 4B, \qquad B = \frac{9}{4}.$$

Setting $x = -3$ gives

$$-11 = -4A, \qquad A = \frac{11}{4}.$$

Therefore,

$$\begin{aligned}\int \frac{5x+4}{x^2+2x-3}\,dx &= \int \left(\frac{11/4}{x+3} + \frac{9/4}{x-1}\right) dx \\ &= \frac{11}{4}\ln|x+3| + \frac{9}{4}\ln|x-1| + C.\end{aligned}$$

6. Evaluate the definite integral:

$$\int_{\pi/6}^{\pi/3} \cot(x)\,dx.$$

Write

$$\cot x = \frac{\cos x}{\sin x}.$$

Let

$$u = \sin x, \qquad du = \cos x\,dx.$$

When $x = \pi/6$, we have $u = 1/2$, and when $x = \pi/3$, we have $u = \sqrt{3}/2$. Thus,

$$\int_{\pi/6}^{\pi/3} \cot(x)\,dx = \int_{1/2}^{\sqrt{3}/2} \frac{1}{u}\,du = \ln|u|\Big|_{1/2}^{\sqrt{3}/2}.$$

Therefore,

$$= \ln\left(\frac{\sqrt{3}}{2}\right) - \ln\left(\frac{1}{2}\right) = \ln(\sqrt{3}).$$

7. Evaluate the indefinite integral:

$$\int \frac{1}{\sqrt{4x-x^2}}\,dx.$$

Complete the square:

$$4x - x^2 = 4 - (x-2)^2.$$

Thus,

$$\int \frac{1}{\sqrt{4x-x^2}}\,dx = \int \frac{1}{\sqrt{4-(x-2)^2}}\,dx.$$

Let

$$u = \frac{x-2}{2}, \qquad dx = 2\,du.$$

Then

$$\sqrt{4-(x-2)^2} = 2\sqrt{1-u^2}.$$

Therefore,

$$\int \frac{1}{\sqrt{4-(x-2)^2}}\,dx = \int \frac{1}{2\sqrt{1-u^2}}(2\,du) = \int \frac{1}{\sqrt{1-u^2}}\,du.$$

Hence,

$$= \arcsin(u) + C = \arcsin\left(\frac{x-2}{2}\right) + C.$$

8. Evaluate the definite integral:

$$\int_1^{e^2} \frac{1}{x\sqrt{\ln x}}\,dx.$$

Let

$$u = \ln x, \qquad du = \frac{1}{x}\,dx.$$

When $x = 1$, we have $u = 0$, and when $x = e^2$, we have $u = 2$. Thus,

$$\int_1^{e^2} \frac{1}{x\sqrt{\ln x}}\,dx = \int_0^2 u^{-1/2}\,du = 2u^{1/2}\Big|_0^2.$$

Therefore,

$$= 2\sqrt{2}.$$

9. Let $f(x) = x^2$ on $[0, 2]$.

   (a) Write the right-endpoint Riemann sum with n subintervals.
   We have

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}, \qquad x_i = 0 + i\Delta x = \frac{2i}{n}.$$

   Thus, the right-endpoint Riemann sum is

$$\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^2 \frac{2}{n} = \frac{8}{n^3}\sum_{i=1}^{n} i^2.$$

   (b) Evaluate the limit of the sum as $n \to \infty$.
   Using

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6},$$

   we get

$$\frac{8}{n^3}\sum_{i=1}^{n} i^2 = \frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{4(n+1)(2n+1)}{3n^2}.$$

   Therefore,

$$\lim_{n\to\infty} \frac{4(n+1)(2n+1)}{3n^2} = \frac{8}{3}.$$

(c) Use the Fundamental Theorem of Calculus to verify your answer.

Since an antiderivative of x^2 is

$$F(x) = \frac{x^3}{3},$$

the Fundamental Theorem of Calculus gives

$$\int_0^2 x^2\,dx = \frac{x^3}{3}\Big|_0^2 = \frac{8}{3}.$$

This agrees with the limit of the Riemann sums.

10. Let $F(x)$ be an antiderivative of $f(x)$, and let $g(x)$ be a differentiable function.

(a) Show that $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$.

Since $F'(x) = f(x)$, the chain rule gives

$$\frac{d}{dx}F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x).$$

Thus, $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$.

(b) Use this result to explain why the substitution $u = g(x)$ works when evaluating integrals.

Since $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$, we have

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C.$$

If we let

$$u = g(x),$$

then this becomes

$$\int f(u)\,du = F(u) + C.$$

Substituting back $u = g(x)$ gives

$$F(g(x)) + C.$$

This explains why substitution works.

11. Let $f(x) = F'(x)$ be continuous on $[a, b]$.

(a) Use the Mean Value Theorem to show that for each subinterval $[x_{i-1}, x_i]$, there exists $c_i \in [x_{i-1}, x_i]$ such that

$$f(c_i)\Delta x = F(x_i) - F(x_{i-1}).$$

Since F is differentiable on (x_{i-1}, x_i) and continuous on $[x_{i-1}, x_i]$, the Mean Value Theorem guarantees a number

$$c_i \in (x_{i-1}, x_i)$$

such that

$$F'(c_i) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}.$$

Because $F'(x) = f(x)$ and $x_i - x_{i-1} = \Delta x$, we obtain

$$f(c_i) = \frac{F(x_i) - F(x_{i-1})}{\Delta x}.$$

Multiplying by Δx gives

$$f(c_i)\Delta x = F(x_i) - F(x_{i-1}).$$

(b) Write the Riemann sum

$$\sum_{i=1}^{n} f(c_i)\Delta x$$

using this result.

Using part (a),

$$\sum_{i=1}^{n} f(c_i)\Delta x = \sum_{i=1}^{n} \big(F(x_i) - F(x_{i-1})\big).$$

(c) Explain why the sum telescopes.

Expanding the sum gives

$$(F(x_1) - F(x_0)) + (F(x_2) - F(x_1)) + \cdots + (F(x_n) - F(x_{n-1})).$$

All middle terms cancel, leaving

$$F(x_n) - F(x_0) = F(b) - F(a).$$

(d) Use this to explain why

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Since

$$\sum_{i=1}^{n} f(c_i)\Delta x = F(b) - F(a)$$

for every n, taking the limit as $n \to \infty$ gives

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

This is the Fundamental Theorem of Calculus.

12. Consider the function $f(x) = \frac{1}{1+x^2}$ on $[a, b]$.

   (a) Show that there exists $c_i \in [x_{i-1}, x_i]$ such that

$$f(c_i)\Delta x = \arctan(x_i) - \arctan(x_{i-1}).$$

   Let

$$F(x) = \arctan(x).$$

   Then

$$F'(x) = \frac{1}{1+x^2} = f(x).$$

   By the Mean Value Theorem, for each subinterval $[x_{i-1}, x_i]$ there exists

$$c_i \in (x_{i-1}, x_i)$$

   such that

$$F'(c_i) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}.$$

   Since $x_i - x_{i-1} = \Delta x$, this becomes

$$\frac{1}{1+c_i^2} = \frac{\arctan(x_i) - \arctan(x_{i-1})}{\Delta x}.$$

   Multiplying by Δx gives

$$f(c_i)\Delta x = \arctan(x_i) - \arctan(x_{i-1}).$$

   (b) Solve for c_i in terms of x_{i-1} and x_i.

   From part (a),

$$\frac{\Delta x}{1+c_i^2} = \arctan(x_i) - \arctan(x_{i-1}).$$

   Thus,

$$1 + c_i^2 = \frac{\Delta x}{\arctan(x_i) - \arctan(x_{i-1})},$$

   so

$$c_i^2 = \frac{\Delta x}{\arctan(x_i) - \arctan(x_{i-1})} - 1.$$

   Hence,

$$c_i = \sqrt{\frac{\Delta x}{\arctan(x_i) - \arctan(x_{i-1})} - 1}.$$

   Since c_i is guaranteed by the Mean Value Theorem to lie in $[x_{i-1}, x_i]$, this is the desired expression.

(c) Use this to show that the Riemann sum telescopes.

Using part (a),

$$\sum_{i=1}^{n} f(c_i)\Delta x = \sum_{i=1}^{n} \big(\arctan(x_i) - \arctan(x_{i-1}) \big).$$

Expanding gives

$$(\arctan(x_1)-\arctan(x_0))+(\arctan(x_2)-\arctan(x_1))+\cdots+(\arctan(x_n)-\arctan(x_{n-1})),$$

so all middle terms cancel. Therefore,

$$\sum_{i=1}^{n} f(c_i)\Delta x = \arctan(x_n) - \arctan(x_0) = \arctan(b) - \arctan(a).$$

(d) Conclude that

$$\int_a^b \frac{1}{1+x^2}\, dx = \arctan(b) - \arctan(a).$$

Taking the limit of the Riemann sums gives

$$\int_a^b \frac{1}{1+x^2}\, dx = \arctan(b) - \arctan(a).$$