Math 140: Calculus I (Spring 2026)
Homework Week 12 Solutions

Relevant Topics: antiderivatives, indefinite integrals, definite integrals, area, sequences of partial sums

1. Evaluate the indefinite integral:

$$\int \left(6x^5 - 4x^2 + 3\right)\, dx$$

Using the power rule,

$$\int \left(6x^5 - 4x^2 + 3\right)\, dx = x^6 - \frac{4}{3}x^3 + 3x + C.$$

$$\boxed{\int \left(6x^5 - 4x^2 + 3\right)\, dx = x^6 - \frac{4}{3}x^3 + 3x + C}$$

2. Evaluate the indefinite integral:

$$\int \left(\frac{1}{x} + \sec^2 x - e^x\right) dx$$

Using basic antiderivative formulas,

$$\int \left(\frac{1}{x} + \sec^2 x - e^x\right) dx = \ln|x| + \tan x - e^x + C.$$

$$\boxed{\int \left(\frac{1}{x} + \sec^2 x - e^x\right) dx = \ln|x| + \tan x - e^x + C}$$

3. Evaluate the indefinite integral:

$$\int \left(3x^{1/2} + 2x^{-2} - 5\cos x\right) dx$$

Using the power rule and trig formulas,

$$\int 3x^{1/2}\, dx = 3 \cdot \frac{x^{3/2}}{3/2} = 2x^{3/2},$$

$$\int 2x^{-2}\, dx = 2 \cdot \frac{x^{-1}}{-1} = -\frac{2}{x},$$

$$\int -5\cos x\,dx = -5\sin x.$$

Thus,

$$\boxed{\int \left(3x^{1/2} + 2x^{-2} - 5\cos x\right) dx = 2x^{3/2} - \frac{2}{x} - 5\sin x + C}$$

4. Evaluate the indefinite integral:

$$\int \left(4x^3 - \frac{2}{x} + 3e^x + \frac{1}{1+x^2}\right) dx$$

Term-by-term,

$$\int 4x^3\,dx = x^4, \qquad \int -\frac{2}{x}\,dx = -2\ln|x|,$$
$$\int 3e^x\,dx = 3e^x, \qquad \int \frac{1}{1+x^2}\,dx = \arctan x.$$

Therefore,

$$\boxed{\int \left(4x^3 - \frac{2}{x} + 3e^x + \frac{1}{1+x^2}\right) dx = x^4 - 2\ln|x| + 3e^x + \arctan x + C}$$

5. Evaluate the definite integral using geometry:

$$\int_0^6 5dx$$

This is the area of a rectangle with base 6 and height 5:

$$\int_0^6 5\,dx = 6 \cdot 5 = 30.$$

$$\boxed{30}$$

6. Evaluate the definite integral using geometry:

$$\int_0^4 xdx$$

This is the area of a right triangle with base 4 and height 4:

$$\int_0^4 x\,dx = \frac{1}{2}(4)(4) = 8.$$

$$\boxed{8}$$

7. Evaluate the definite integral using geometry:

$$\int_{-1}^1 \sqrt{1-x^2}dx$$

The graph of $y = \sqrt{1-x^2}$ is the upper semicircle of radius 1. Therefore,

$$\int_{-1}^1 \sqrt{1-x^2}\,dx = \frac{1}{2}\pi(1)^2 = \frac{\pi}{2}.$$

$$\boxed{\frac{\pi}{2}}$$

8. Evaluate the definite integral using geometry:

$$\int_0^3 (x+1)dx$$

The region under $y = x + 1$ from $x = 0$ to $x = 3$ is a trapezoid with bases 1 and 4 and height 3. Hence,

$$\int_0^3 (x+1)\,dx = \frac{1}{2}(1+4)(3) = \frac{15}{2}.$$

$$\boxed{\frac{15}{2}}$$

9. Let

$$A_n^{\text{right}}$$

denote the right-endpoint rectangle approximation to

$$\int_{-1}^2 x^2 dx$$

using n subintervals of equal width.

Since the interval $[-1, 2]$ has length 3,

$$\Delta x = \frac{3}{n}.$$

The right endpoints are

$$x_i = -1 + \frac{3i}{n}, \qquad i = 1, 2, \ldots, n.$$

(a) Compute A_4^{right}.

Here

$$\Delta x = \frac{3}{4},$$

and the right endpoints are

$$-\frac{1}{4}, \quad \frac{1}{2}, \quad \frac{5}{4}, \quad 2.$$

Thus,

$$A_4^{\text{right}} = \frac{3}{4}\left[\left(-\frac{1}{4}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{5}{4}\right)^2 + 2^2\right]$$

$$= \frac{3}{4}\left(\frac{1}{16} + \frac{1}{4} + \frac{25}{16} + 4\right) = \frac{3}{4}\left(\frac{94}{16}\right) = \frac{141}{32}.$$

$$\boxed{A_4^{\text{right}} = \frac{141}{32}}$$

(b) Write A_n^{right} as a summation.

$$A_n^{\text{right}} = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \left(-1 + \frac{3i}{n}\right)^2 \frac{3}{n}.$$

$$\boxed{A_n^{\text{right}} = \frac{3}{n}\sum_{i=1}^{n} \left(-1 + \frac{3i}{n}\right)^2}$$

(c) Use summation formulas to write A_n^{right} without sigma notation.
First expand:

$$\left(-1 + \frac{3i}{n}\right)^2 = 1 - \frac{6i}{n} + \frac{9i^2}{n^2}.$$

So

$$\begin{aligned} A_n^{\text{right}} &= \frac{3}{n}\sum_{i=1}^{n} \left(1 - \frac{6i}{n} + \frac{9i^2}{n^2}\right) \\ &= \frac{3}{n}\left(\sum_{i=1}^{n} 1 - \frac{6}{n}\sum_{i=1}^{n} i + \frac{9}{n^2}\sum_{i=1}^{n} i^2\right). \end{aligned}$$

Using

$$\sum_{i=1}^{n} 1 = n, \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6},$$

we obtain

$$\begin{aligned} A_n^{\text{right}} &= \frac{3}{n}\left(n - \frac{6}{n}\cdot\frac{n(n+1)}{2} + \frac{9}{n^2}\cdot\frac{n(n+1)(2n+1)}{6}\right) \\ &= \frac{3}{n}\left(n - 3(n+1) + \frac{3}{2}\cdot\frac{(n+1)(2n+1)}{n}\right). \end{aligned}$$

Simplifying,

$$A_n^{\text{right}} = 3 + \frac{9}{2n} + \frac{9}{2n^2}.$$

$$\boxed{A_n^{\text{right}} = 3 + \frac{9}{2n} + \frac{9}{2n^2}}$$

(d) Evaluate $\lim_{n\to\infty} A_n^{\text{right}}$.

$$\lim_{n\to\infty} A_n^{\text{right}} = \lim_{n\to\infty}\left(3 + \frac{9}{2n} + \frac{9}{2n^2}\right) = 3.$$

$$\boxed{\lim_{n\to\infty} A_n^{\text{right}} = 3}$$

10. Let

$$A_n^{\text{left}}$$

denote the left-endpoint rectangle approximation to

$$\int_{-1}^{2} x^2 dx$$

using n subintervals of equal width.

The left endpoints are

$$x_i = -1 + \frac{3(i-1)}{n}, \qquad i = 1, 2, \ldots, n.$$

(a) Compute A_4^{left}.

Here

$$\Delta x = \frac{3}{4},$$

and the left endpoints are

$$-1, \quad -\frac{1}{4}, \quad \frac{1}{2}, \quad \frac{5}{4}.$$

Thus,

$$A_4^{\text{left}} = \frac{3}{4}\left[1 + \left(-\frac{1}{4}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{5}{4}\right)^2\right]$$

$$= \frac{3}{4}\left(1 + \frac{1}{16} + \frac{1}{4} + \frac{25}{16}\right) = \frac{3}{4}\left(\frac{46}{16}\right) = \frac{69}{32}.$$

$$\boxed{A_4^{\text{left}} = \frac{69}{32}}$$

(b) Write A_n^{left} as a summation.

$$A_n^{\text{left}} = \sum_{i=1}^{n}\left(-1 + \frac{3(i-1)}{n}\right)^2 \frac{3}{n}.$$

$$\boxed{A_n^{\text{left}} = \frac{3}{n}\sum_{i=1}^{n}\left(-1 + \frac{3(i-1)}{n}\right)^2}$$

(c) Use summation formulas to write A_n^{left} without sigma notation.

Expand:

$$\left(-1 + \frac{3(i-1)}{n}\right)^2 = 1 - \frac{6(i-1)}{n} + \frac{9(i-1)^2}{n^2}.$$

Hence,

$$A_n^{\text{left}} = \frac{3}{n}\sum_{i=1}^{n}\left(1 - \frac{6(i-1)}{n} + \frac{9(i-1)^2}{n^2}\right).$$

Now use

$$\sum_{i=1}^{n} 1 = n, \qquad \sum_{i=1}^{n}(i-1) = \sum_{j=0}^{n-1} j = \frac{n(n-1)}{2},$$

$$\sum_{i=1}^{n}(i-1)^2 = \sum_{j=0}^{n-1} j^2 = \frac{(n-1)n(2n-1)}{6}.$$

So

$$A_n^{\text{left}} = \frac{3}{n}\left(n - \frac{6}{n}\cdot\frac{n(n-1)}{2} + \frac{9}{n^2}\cdot\frac{(n-1)n(2n-1)}{6}\right).$$

Simplifying,

$$A_n^{\text{left}} = 3 - \frac{9}{2n} + \frac{9}{2n^2}.$$

$$\boxed{A_n^{\text{left}} = 3 - \frac{9}{2n} + \frac{9}{2n^2}}$$

(d) Evaluate $\lim_{n\to\infty} A_n^{\text{left}}$.

$$\lim_{n\to\infty} A_n^{\text{left}} = \lim_{n\to\infty}\left(3 - \frac{9}{2n} + \frac{9}{2n^2}\right) = 3.$$

$$\boxed{\lim_{n\to\infty} A_n^{\text{left}} = 3}$$

11. Let

$$A_n^{\text{mid}}$$

denote the midpoint rectangle approximation to

$$\int_{-1}^{2} x^2 dx$$

using n subintervals of equal width.

The midpoints are

$$x_i = -1 + \frac{3(2i-1)}{2n}, \qquad i = 1, 2, \ldots, n.$$

(a) Compute A_4^{mid}.

Here

$$\Delta x = \frac{3}{4},$$

and the midpoints are

$$-\frac{5}{8}, \quad \frac{1}{8}, \quad \frac{7}{8}, \quad \frac{13}{8}.$$

Thus,

$$A_4^{\text{mid}} = \frac{3}{4}\left[\left(-\frac{5}{8}\right)^2 + \left(\frac{1}{8}\right)^2 + \left(\frac{7}{8}\right)^2 + \left(\frac{13}{8}\right)^2\right]$$

$$= \frac{3}{4}\left(\frac{25}{64} + \frac{1}{64} + \frac{49}{64} + \frac{169}{64}\right) = \frac{3}{4}\cdot\frac{244}{64} = \frac{183}{64}.$$

$$\boxed{A_4^{\text{mid}} = \frac{183}{64}}$$

(b) Write A_n^{mid} as a summation.

$$A_n^{\text{mid}} = \sum_{i=1}^{n}\left(-1 + \frac{3(2i-1)}{2n}\right)^2 \frac{3}{n}.$$

$$\boxed{A_n^{\text{mid}} = \frac{3}{n}\sum_{i=1}^{n}\left(-1 + \frac{3(2i-1)}{2n}\right)^2}$$

(c) Use summation formulas to write A_n^{mid} without sigma notation.

First write

$$-1 + \frac{3(2i-1)}{2n} = \frac{6i - (2n+3)}{2n}.$$

So

$$A_n^{\text{mid}} = \frac{3}{n}\sum_{i=1}^{n}\left(\frac{6i-(2n+3)}{2n}\right)^2 = \frac{3}{4n^3}\sum_{i=1}^{n}\left(6i-(2n+3)\right)^2.$$

Expand:

$$(6i-(2n+3))^2 = 36i^2 - 12(2n+3)i + (2n+3)^2.$$

Hence

$$A_n^{\text{mid}} = \frac{3}{4n^3}\left(36\sum_{i=1}^{n} i^2 - 12(2n+3)\sum_{i=1}^{n} i + \sum_{i=1}^{n}(2n+3)^2\right).$$

Using the summation formulas,

$$A_n^{\text{mid}} = \frac{3}{4n^3}\left(36\cdot\frac{n(n+1)(2n+1)}{6} - 12(2n+3)\cdot\frac{n(n+1)}{2} + n(2n+3)^2\right).$$

Simplifying,

$$A_n^{\text{mid}} = 3 - \frac{9}{4n^2}.$$

$$\boxed{A_n^{\text{mid}} = 3 - \frac{9}{4n^2}}$$

(d) Evaluate $\lim_{n\to\infty} A_n^{\text{mid}}$.

$$\lim_{n\to\infty} A_n^{\text{mid}} = \lim_{n\to\infty}\left(3 - \frac{9}{4n^2}\right) = 3.$$

$$\boxed{\lim_{n\to\infty} A_n^{\text{mid}} = 3}$$

12. **Challenge.** Derive a sequence of partial sums that converges to π.

Consider the semicircle

$$y = \sqrt{1-x^2}, \qquad -1 \le x \le 1.$$

Its area is

$$\int_{-1}^{1} \sqrt{1-x^2}\,dx = \frac{\pi}{2}.$$

Divide $[-1,1]$ into n equal subintervals. Then

$$\Delta x = \frac{2}{n},$$

and using right endpoints,

$$x_i = -1 + \frac{2i}{n}.$$

A right-endpoint approximation is

$$A_n = \frac{2}{n}\sum_{i=1}^{n}\sqrt{1-\left(-1+\frac{2i}{n}\right)^2}.$$

As $n \to \infty$,

$$A_n \to \frac{\pi}{2}.$$

Therefore,

$$\boxed{\pi = \lim_{n\to\infty} \frac{4}{n}\sum_{i=1}^{n}\sqrt{1-\left(-1+\frac{2i}{n}\right)^2}}.$$

So one sequence of partial sums converging to π is

$$\boxed{P_n = \frac{4}{n}\sum_{i=1}^{n}\sqrt{1-\left(-1+\frac{2i}{n}\right)^2}}.$$